

AFOSR-TR- 80-0877

31 August 1980
*An Analysis of an Inventory System
for Interchangeable Recoverable Items*

Final

During the period of the grant Professors David Heath, John Muckstadt, and Carol Shilepsky have conducted research on the interchangeability/substitutability problem for recoverable items (items that are subject to repair when they fail). This problem arises when recoverable items can be substituted for one another during the repair of an assembly.

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AFOSR-78-3568

The long range objective of the research project is to develop a method that the Air Force can use to assist in the management of interchangeable recoverable items. To accomplish this goal we have proposed to analyze several simplified problems that will give us insight into the form of the optimal or near-optimal policy for the real problem. Specifically, we proposed:

- (a) to understand fully the behavior of a single-echelon, two-item system over an infinite horizon when the failure processes for the items are independent, stationary Poisson processes and the repair times are exponential;
- (b) to develop methods for finding optimal and near-optimal policies for the situation described in (a);
- (c) to extend the results for topics (a) and (b) to systems having many items;
- (d) to extend the analysis to situations in which there are two echelons (depot-base structure) and many items where, as before, the failure processes for the items are independent, stationary Poisson processes and repair times are exponential;
- (e) to develop methods for finding the optimal or near-optimal operating policy for the situation described in (d);

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20. ABSTRACT (Continue on reverse side if necessary and identify by block number) During the period of the grant the interchangeability/substitutability problem for recoverable items was studied. The objective of the grant was to develop methods for managing interchangeable recoverable items. Research conducted under the grant has led to a full understanding of the behavior of a single-echelon, two-item system over an infinite horizon when the failure times are independent, stationary Poisson processes and the repair times are exponential. Approximately optimal policies for managing this,		

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→ system have been found. These methods have been extended to methods for finding approximately optimal policies for managing the single-echelon system with several interchangeable recoverable items.

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- (f) to study the problem when failure and repair processes need not be stationary and the time horizon is finite for both single- and two-echelon systems; and
- (g) to develop heuristic dispatching rules for the dynamic environment described in (f).

To date we have completed objects (a) and (b), developing heuristics to find a near-optimal policy for the single-echelon, two-item system. We have almost completed objective (c), developing two heuristics which should produce a near-optimal policy for the system having many items. We have begun working towards objectives (d) and (e), establishing the framework and the procedure which will be followed in analyzing the two-echelon system. Details of the progress made towards reaching these objectives are given below.

In meeting objectives (a) and (b), we studied the interchangeability/substitutability problem for two items that fail at a single location. The failure processes for these items are assumed to be independent, stationary Poisson processes, and repair times are exponentially distributed. The system studied is assumed to be a closed system; that is, no items are added to or deleted from the system. Based on these assumptions we first showed that the problem could be viewed as a Markovian decision problem for which there exists a stationary optimal policy. Since our main goal is to find methods that can be used to solve the real problem, we next developed various approaches for finding optimal and near-optimal policies.

The first approach we took was to formulate the decision problem as a linear programming problem. The number of states in the Markovian

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decision problem is so large that the linear programming method would not be a practical method for finding optimal policies. We developed a procedure for avoiding this difficulty. We found that the behavior of a system with many parts could be satisfactorily studied by considering a smaller system with fewer parts, and thus fewer states in its associated Markovian decision problem. The number of states can be further reduced by eliminating certain unlikely states from consideration. With fewer states, it becomes practical to use linear programming to obtain an optimal solution for the smaller system. This solution can be translated into a near-optimal solution for the original system. Thus we can use linear programming to obtain near-optimal solutions for the single-location, two-item system.

Next, we studied the solutions obtained by linear programming and were able to identify some properties that an optimal policy should possess. We developed a simulation approach that exploits these properties that an optimal policy is conjectured to have. This heuristic method is computationally efficient and finds, at least for the cases tested, a nearly optimal policy. (The details of the results of this work can be found in the attached technical report, entitled "An Analysis of a Single Location Inventory Problem for Two Interchangeable Recoverable Items.")

In working towards objective (c), we extended the system studied in (a) and (b) to include several interchangeable recoverable items. We found that the number of states and possible actions grew so huge that any attempt to use linear programming to find an optimal policy would be impractical. Hence we turned our attention to the development of heuristic methods which should produce a good policy.

Our first approach involved regarding the multiple-item system as a sequence of two-item systems. At each stage in the sequence the linear programming procedure developed for objectives (a) and (b) is used to find a near-optimal policy for a segment of the multiple-item system. This method produces a rule prescribing when a substitution is to be made. (The details involved in calculating and implementing this substitution rule can be found in the attached Technical Report, entitled "A Procedure for Finding a Nearly Optimal Policy for the Inventory System with Several Interchangeable Recoverable Items.")

Our second approach involved formulating the multiple-item decision problem as a policy improvement problem. Beginning with the policy of never making any substitutions, the special structure of this problem can be exploited to permit us to carry out the policy improvement algorithm for a single iteration, producing an improved substitution policy. Preliminary computational experience indicates that this policy is a good one.

In beginning to work towards objectives (d) and (e), we have established the framework for studying the two-echelon system. We are constructing a flexible simulation model that will enable us to evaluate and compare the performance of several alternative depot and base decision rules. This simulation will also be used to evaluate the performance of the two heuristic rules developed under objective (c).

We have presented the results of our research at the International Meeting of the Institute of Management Sciences in Hawaii in June 1979, at the Multi-Level Inventory Conference in Philadelphia in October 1979,

and at the Annual Meeting of the Operations Research Society of America in Washington, D.C., in May 1980. Furthermore, we plan to prepare a paper having the same title as the first attached technical report and submit it for publication in the Naval Research Logistics Quarterly..

Throughout our investigations we have been in contact with Lt. Col. Jon Hobbs and Mr. Victor Presutti of Headquarters, Air Force Logistics Command (AFLC/XRS). Their guidance and assistance in providing experimental data has been extremely valuable.

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Almost sure convergence of uniform transport processes to Brownian motion (with R. Griego and A. Ruiz-Moncayo), Ann. Math. Stat. 42 (1971), 1129-1131.

Subfair Red-and-Black with a limit (with W.E. Pruitt and W.D. Sudderth), Proc. Amer. Math. Soc. 35 (1972), 555-560.

On a theorem of de Finetti, oddsmaking, and game theory (with W.D. Sudderth), Ann. Math. Stat. 43 (1972), 2072-2077.

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Red-and-Black with unknown win probability (with D. Berry and W.D. Sudderth), Ann. Stat. 2 (1974), 602-608.

Skorokhod stopping via potential theory, Seminaire de Prob. VIII, Lecture Notes in Math. 381 (1974), 150-154.

Continuous time gambling problems (with W.D. Sudderth), Adv. Appl. Prob. 6 (1974), 651-665.

Skorokhod stopping in discrete time, Seminaire de Prob. IX, Lecture Notes in Math. 465 (1975), 515-517.

Interpolation of martingales, Ann. Prob. 5 (1977), 804-806.

On exchangeable random variables (with W.D. Sudderth), American Statistician 30 (1976), 188-189.

Algorithms for computation of blackjack strategies, presented at Second Annual Conference on Gambling, Lake Tahoe, Nevada, 1975.

On finitely additive priors, coherence, and extended admissibility (with W.D. Sudderth), Ann. Stat. 6 (1978), 333-345.

Internal telephone billing rates - a novel application of non-atomic game theory (with L.J. Billera and J. Raanan), Operations Research 26 (1978), 956-965.

Contact stress and wear study for type characters (with F.I. Chow, P.A. Engel, and S. Lawphongpanich), IBM Journal of Research and Development 22 (1978), 658-667.

On inadequacies of countably additive measures (with L. Dubins), In preparation.

An Analysis of a single location inventory problem for two interchangeable recoverable items (with J. Muchstadt and C. Shilepsky), Tech. Report 409, School of ORIE, Cornell University.

On some aspects of pari-mutuel wagering, presented at TIMS-ORSA Joint National Meeting, Washington D.C., 1980.

Allocation of shared costs: A set of axioms yielding a unique procedure (with L.J. Billera), to appear in Math. of O.R.

A unique procedure for allocating common costs from a production process (with L.J. Billera and R. Verecchia), to appear in Journal of Accounting Research.

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"Scheduling in Power Systems." Doctoral Dissertation, College of Engineering, University of Michigan, 1966.

"An Application of Mixed-Integer Programming Duality to Scheduling Thermal Generating Systems." (with Richard C. Wilson) IEEE Transactions, Part III (PAS) December 1968, pp. 1968-1978.

"An Application of Duality Theory to Zero-One Integer Programs Having Convex Objective Functions." AFITSL Technical Report 6-69. Air University, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1969.

"A Dual Decomposition Algorithm for Solving Mixed Integer-Continuous Quadratic Programming Problems." AFITSL Technical Report 7-69. Air University, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1969.

"An Algorithm for Determining Optimal Stock Levels in a Multi-Echelon Inventory System." Air University, Air Force Institute of Technology, Wright-Patterson Air Force Base, Ohio, 1971.

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"NAVMET: A Four-Echelon Model for Determining the Optimal Quantity and Distribution of Navy Spare Aircraft Engines." Report R-7511. Naval Weapons Engineering Support Activity, Washington Navy Yard, Washington, D.C., December 1975.

"An Application of Lagrangian Relaxation to Scheduling in Power Generation Systems." (with Sherri Koenig) Operations Research, Vol. 25, No. 3, May-June 1977, pp. 387-403. (An expanded version of this paper is included in a book entitled Studies in Operations Management edited by A. Hax, North-Holland, Amsterdam, 1978.)

"On the Probability Distribution for Inventory Position in Two-Echelon Continuous Review Systems." School of Operations Research and Industrial Engineering Technical Report No. 336. Cornell University, Ithaca, N.Y., June 1977.

"Analysis of a Two-Echelon Inventory System in Which All Locations Follow Continuous Review (S,s) Policies." School of Operations Research and Industrial Engineering Technical Report No. 337. Cornell University, Ithaca, N.Y., June 1977.

"An Analysis of Two Two-Echelon Inventory Systems." (with Paul Sasseville) School of Operations Research and Industrial Engineering Technical Report No. 361. Cornell University, Ithaca, N.Y., September 1977.

"Optimal Policy for Batch Operations: Backup, Checkpointing, Reorganization, and Updating." (with Guy Lohman) ACM Transactions on Data Base Systems, Vol. 2, No. 3, September 1977.

"Some Approximations in Multi-Item, Multi-Echelon Inventory Systems for Recoverable Items." Naval Research Logistics Quarterly, Vol. 25, No. 3, September 1978, pp. 377-393.

"Comments on 'Single Cycle Continuous Review Policies for Arborvescent Production Inventory Systems'." (with H. Singer) Management Science, Vol. 24, No. 16, December 1978, pp. 1766-1768.

"An Analysis of a Single Location Inventory Problem for Two Interchangeable Recoverable Items." (with D. Heath and C. Shilepsky) School of Operations Research and Industrial Engineering Technical Report No. 409. Cornell University, Ithaca, N.Y., February 1979.

"Coordination of Production Schedules with Shipping Schedules." (with W. L. Maxwell) School of Operations Research and Industrial Engineering Technical Report No. 422. Cornell University, Ithaca, N.Y., April 1979. (Will appear in a book entitled "Multi-Level Production/Inventory Systems: Theory and Practice," which is in the Studies in the Management Sciences series sponsored by TIMS.)

"A Three-Echelon, Multi-Item Model for Recoverable Items." Naval Research Logistics Quarterly, Vol. 26, No. 2, June 1979, pp. 199-221.

"Are Multi-Echelon Inventory Methods Worth Implementing in Systems with Low-Demand Rate Items?" (with L. Joseph Thomas) (Forthcoming in Management Science.)

"Cost Comparisons of Alternative Methods for Managing Multi-Level Inventory Systems: A Case Study." (with L. Joseph Thomas) Tijdschrift voor Economie en Management, Vol. XXV, No. 1, 1980. (This was an invited paper for a special issue on operations management, and is closely related to the preceding paper co-authored with L. J. Thomas.)

"An Analysis of a Single Item Inventory System with Returns: The Single Echelon Case." (with M. Isaac) School of Operations Research and Industrial Engineering Technical Report No. 431. Cornell University, Ithaca, N.Y., July 1979.

"An Analysis of a Single Item Inventory System with Returns: The Multi Echelon Case." (with M. Isaac) School of Operations Research and Industrial Engineering Technical Report No. 432. Cornell University, Ithaca, N.Y., July 1979.

(The preceding two papers have been combined into one which will appear in the Naval Research Logistics Quarterly.)

"Cycling Policies in One Warehouse, N Retailer Production Inventory Systems." (with H. Singer) (Submitted for publication.)

"An Examination of a Two-Echelon Inventory System for Recoverable Items When the Demand Process is Non-Stationary." RAND Corporation, Santa Monica, CA. N-1493-AF. 1980.

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Calculus, Differential Equations, Real and Complex Analysis,
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Research and Publications:

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"A Boundedness Theorem for Solutions of a Volterra Equation," thesis, University of Wisconsin, Madison, WI, 1970.

"A Boundedness Theorem for Volterra Equations," (co-author, Kenneth Hannsgen), Journal of Differential Equations 10 (1971), pp. 378-387.

"A Note on the Asymptotic Behavior of an Integral Equation," Proceedings of the American Mathematical Society 33 (1972), pp. 111-113.

"The Asymptotic Behavior of an Integral Equation with an Application to Volterra's Population Equation," Journal of Mathematical Analysis and Applications, Volume 48, Number 3 (1974), pp. 764-779.

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"Computational Considerations for a Single Location Inventory Problem for Two Interchangeable Recoverable Items," School of Operations Research and Industrial Engineering Technical Report No. 439, Cornell University, Ithaca, N.Y., November 1979.

"Simulation Techniques for an Inventory System with Interchangeable Recoverable items," School of Operations Research and Industrial Engineering Technical Report No. 459, Cornell University, Ithaca, N.Y., May 1980.

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TECHNICAL REPORT NO. 409

February 1979

AN ANALYSIS OF A SINGLE LOCATION INVENTORY
PROBLEM FOR TWO INTERCHANGEABLE RECOVERABLE ITEMS

by

David Heath
John Muckstadt
Carol Shilepsky

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TABLE OF CONTENTS

	Page
I. INTRODUCTION	1
II. A TWO-ITEM SINGLE LOCATION SYSTEM: A DISCRETE TIME MODEL	3
III. A TWO-ITEM SINGLE LOCATION SYSTEM: A CONTINUOUS TIME MODEL	6
IV. THE FORM OF AN OPTIMAL POLICY	13
V. APPROXIMATIONS TO AN OPTIMAL POLICY	15
VI. A SUMMARY AND COMMENTS CONCERNING FUTURE EFFORTS	22
VII. REFERENCES	23

ABSTRACT

In this paper we examine the interchangeability/substitutability problem for two recoverable items that fail at a single location. We assume the failure processes for each type of item are independent, stationary Poisson processes. We also assume the repair times are exponentially distributed. Furthermore, we assume that the system is a closed system, that is, no items are added to or deleted from the system. We first consider a discrete-time problem and show that this problem is a Markovian decision problem. We then show that for this problem there exist optimal stationary Markov control policies. Next we formulate a continuous time model and show how to find the optimal stationary Markov control policy using linear programming. Unfortunately, this approach is impractical for solving most real problems. Consequently we have established and explored some of the properties that we feel an optimal policy should possess. A discussion of these properties is given in Section IV. Lastly, we will describe a heuristic that can be used to find a good policy. This method is an efficient simulation search method that finds policies having the properties we conjecture an optimal policy should possess.

I. INTRODUCTION

During the past 15 years a substantial amount of research has been conducted related to the management of recoverable items, that is, items subject to repair when they fail [1,2,3,4,5,6,7,8,9,11,12,13,14]. A number of mathematical models have been developed that can be used to determine optimal stockage levels for each recoverable item in both single and multi-echelon systems. Most of the models are based on the assumption that the items are independent. That is, the failure processes among the items are assumed to be independent. Some recent research has been devoted to dependencies in the demand process by recognizing that certain recoverable items have a hierarchical design [1,5,6,13,14]. For these items, the failure of a recoverable component results in a demand for both a spare component and the assembly containing the component. However, in all of the models presented to date the replacement rule for a failed component is the same: replace all failed units with a serviceable spare item of the same type.

In this paper we examine a problem that arises when items are sometimes interchangeable or can be substituted for one another during repair. Frequently it is possible to repair a broken assembly using several different types of parts; however, choosing the "correct" part to use to repair the assembly is not based on engineering considerations alone. Using one type of item to complete the repair rather than using a second type of item can cause subsequent parts shortages that can be avoided. This can occur because some items are "more useful" than others. For example, suppose there are only two types of items in the system. The "more useful" item can be used to satisfy a demand for either type of item whereas the "less useful" item can only be used to satisfy demands for its own type

of item. Many such interchangeable/substitutable items are found in the Air Force. This is particularly the case for electronic items. In some instances, newly designed items can be used to replace older units when they fail; however, these older units cannot be used to repair a newer generation of an assembly.

In this paper we will examine the interchangeability/substitutability problem for two items that fail at a single location. We assume the failure processes for each type of item are independent, stationary Poisson processes. We also assume the repair times are exponentially distributed. Furthermore, we assume that the system is a closed system, that is, no items are added to or deleted from the system. We first consider a discrete-time problem and show that this problem is a Markovian decision problem. We then show that for this problem there exist optimal stationary Markov control policies. Next we formulate a continuous-time model and show how to find the optimal stationary Markov control policy using linear programming. Unfortunately, this approach is impractical for solving most real problems. Consequently we have established and explored some of the properties that we feel an optimal policy should possess. A discussion of these properties is given in Section IV. Lastly, we will describe a heuristic that can be used to find a good policy. This method is an efficient simulation search method that finds policies having the properties we conjecture an optimal policy should possess.

II. A TWO-ITEM SINGLE LOCATION SYSTEM: A DISCRETE TIME MODEL

In this section we present a discrete-time model for the interchangeability/substitutability problem. We examine this discrete-time model since it yields a particular form for the optimal control policy. In particular, we show that for the associated Markov decision problem there is a stationary Markov control policy which achieves the lowest average back-order level. This result provides the motivation for restricting attention to Markov control policies in the continuous time model developed in the next section.

To simplify the analysis and discussion we restrict our attention to a single location system with only two types of items: type 1 and type 2. The assemblies in which these items are installed are called units; we also assume that there are two types of units and that each unit contains only one item of the types considered. Furthermore, when a type 1 unit or a type 2 unit fails, we assume that it can be repaired with a serviceable type 2 item. Type 1 items can also be used to repair failed type 1 units; however, type 1 items cannot be used to repair type 2 units. For example, two units might be different "generations" of a computer found in a fire control system; the items might be old and new versions of an integrated circuit board found in the computer. The newer version of the circuit board can be used in both generations of the fire control system computer; but, the old generation circuit board is incompatible with the newer fire control system computer.

Let N_i be the number of units of type i and M_i be the number of spare items of type i in the system. Thus there are a total of $N_i + M_i$ items in the system. Let n_{ij} be the number of type i items

installed in type j units, and m_i be the number of servicable type i items in spare stock.

Note that according to the substitution rules we have established n_{12} is always zero. Thus the five numbers m_1 , m_2 , n_{11} , n_{21} , and n_{22} specify the disposition of all items. The number of type i items in repair is given by $N_i + M_i - (m_i + \sum_{j=1}^i n_{ij})$, and the number of backorders associated with type j units is $N_j - \sum_{i=j}^2 n_{ij}$.

We assume the system operates as follows in the discrete time model. At each time $t = n \cdot (\Delta t)$, $n = 0, 1, \dots$ certain actions are available. These actions correspond to installing some items currently in spare stock in appropriate units lacking an item. After installation of items, failures may occur. We presume that items fail independently of one another and that each item of type i installed in a unit of type j has probability $\lambda_{ij} \cdot (\Delta t)$ of failing (where (Δt) is small enough so that these numbers do not exceed 1). Next, items which have failed are removed from the units and sent to repair. Items are repaired independently; we presume that each item is repaired during this time interval with probability $r_i \cdot (\Delta t)$.

After this sequence of action-failure-repair we begin again at the new time $(n+1) \cdot \Delta t$ by selecting another action. The system continues to operate in this manner for an indefinite length of time.

The number of backorders during the action-failure-repair cycle is defined to be the number of backorders which exist immediately after the action (unit repair) is taken and before the failures occur.

We wish to choose those actions which minimize the average number of backorders.

The selection of the particular sequencing of events - action, failure repair - was not made arbitrarily. This sequence was selected so that the problem could be formulated as a Markovian decision problem (average cost model). (A discussion of Markovian decision problems can be found in reference 10.)

Notice that for any policy (that is, a specification of the actions to be taken for all possible situations) there is positive probability (actually bounded away from zero) that after one cycle of action-failure-repair all items which were in use will have failed and been repaired. Hence there is a state (namely that with no items installed and all in spare stock) for which every action taken at every state gets to that state in one step with probability greater than or equal to $\beta > 0$. By Ross [10] (Theorem 6.17, Corollary 6.20, and the remarks following Corollary 6.20) there is then an optimal stationary Markov control policy. This policy can be computed by a technique involving linear programming; this technique is adapted to the continuous-time model developed in Section III.

III. A TWO-ITEM SINGLE LOCATION SYSTEM: A CONTINUOUS TIME MODEL

We now consider the continuous time model corresponding to that of Section II. The notation describing the numbers of units and items and the state of the system remain the same.

We now suppose that the failure times of type i items installed in type j units are independent exponentially-distributed random variables with mean $1/\lambda_{ij}$, and that the repair times are independent, exponential random variables with mean $1/r_i$. As before, the measure of performance of the system is the average number of backorders. Motivated by the results of the previous section we shall consider only stationary Markov control policies.

Since it would seem unreasonable to allow backorders for type 1 units if there are type 1 items in spare stock we consider the installation of type 1 items in type 1 units to be automatic and not subject to control. Thus the actions available involve only the installation of type 2 items in type 1 units. We shall compute the optimal stationary Markov control policy which takes action only when the system changes state due to an item failing in service or being returned from repair. Finally, we allow only the substitution of one type 2 item into type 1 units at a time.

When the process jumps to a new state there are (possibly) two actions available: do nothing, or install a type 2 item in a type 1 unit. Of course, if there are no type 2 items available or no backorders associated with type 1 units, then there is only one action available: do nothing.

Following Ross [10] we allow randomized actions; thus corresponding to state S there are two numbers $P_S^{a_1}$ and $P_S^{a_2}$ (non-negative and summing to 1) giving the probabilities of selecting action a_1 (= do nothing) or a_2 (= put one type 2 item in a type 1 unit). These P 's completely specify the control policy.

Once these P 's are specified the process which results is a stationary Markov process. In fact, if we consider the process which specifies the state the system most recently jumped to and the action taken there, this process is a Markov chain and we can compute its stationary transition probabilities and then the average cost.

Let Z_S^a be the equilibrium (or stationary) probability that this process most recently jumped to state S and that the action taken there was a . Then we must have $\sum_a \sum_{S_2} \alpha_{S_1 S_2}^a Z_{S_2}^a = \sum_a Z_{S_1}^a$ where $\alpha_{S_1 S_2}^a$ ($S_1 \neq S_2$) is the rate at which transitions to state S_1 will occur if the current state is S_2 and the action taken is " a ", and $\alpha_{S_2 S_2}^a = -\sum_{S_1} \alpha_{S_1 S_2}^a$ so that the column-sums ($S_1 \neq S_2$)

of the α -matrix are zero. (The subscripts may seem reversed in the above. This is because in the usual Markov chain matrix notation the order of vector-matrix multiplication is the reverse of that used in the usual L.P. notation, which we adopt here.)

The cost associated with each state, $C(S,a)$, is the number of back-orders (total for units of type 1 and type 2) associated with state S if action a is chosen.

We wish to minimize the cost-per-unit-time given by $\sum_S \sum_a Z_S^a C(S,a)$ subject to the equilibrium equations $\sum_a \sum_{S_2} \alpha_{S_1 S_2}^a Z_{S_2}^a = \sum_a Z_{S_1}^a$ and to the condition $\sum_S \sum_a Z_S^a = 1$

$$Z_S^a \geq 0.$$

(Note that in some states substitution obviously cannot be performed: for these states we can simply ignore (that is set to zero) the corresponding Z_1^a for $a = \text{SUBSTITUTE}$).

The solution to the above stated L.P. provides the values for the P 's (and hence the policy) as follows:

$$p_S^a = P\{\text{take action } a \text{ when in state } S\} \\ = \frac{z_S^a}{\sum_b z_S^b},$$

whenever $\sum_b z_S^b \neq 0$. When $\sum_b z_S^b = 0$, state i is never reached by the controlled system and thus it makes sense to leave the action chosen there undefined.

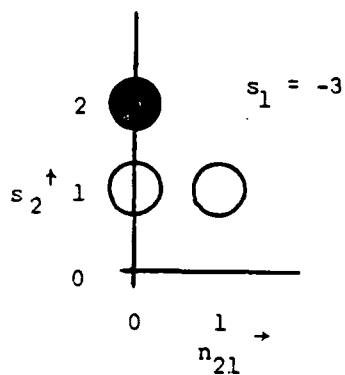
Suppose T represents the number of realizable states in the system (a state is realizable whenever $\sum_a z_S^a > 0$). Then the above optimization problem has T equality constraints and T basic variables (note there is one redundant constraint). If state S is realizable, then $z_S^a > 0$ for at least one action a . Consequently at least one z_S^a is positive for $S=1, \dots, T$. This implies that for each state S , z_S^a can be positive for only one action a . Thus p_S^a must be either 0 or 1.

A program was written to generate the input for a linear programming package (MPSX). The output was then submitted to MPSX and (in almost all cases) an optimal strategy was obtained.

To present the results of the computations we notice that under our control policies the number of variables necessary to describe the state of the system can be reduced from the five above to three as follows: Since we never allow backorders for type i units if there is serviceable spare stock on hand; $N_i - \sum_{j=1}^2 n_{ji}$ and m_i cannot both be positive. We thus set $s_i = m_i - [N_i - \sum_{j=1}^2 n_{ji}]$, which is the net inventory of items of type i (it is negative if there are backorders associated with type i units).

The variables s_1 , s_2 and n_{21} then describe the state of the system.

Finally, since at each state the control policy merely specifies "substitute" or "don't substitute" it suffices to graph the set of states S at which one would perform a substitution. Graphs of the optimal "substitution set" S for several situations are given in Figure 1, Figure 2, and Figure 3.



$$N_1 = 5$$

$$M_1 = 2$$

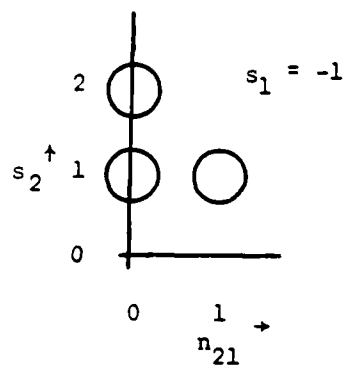
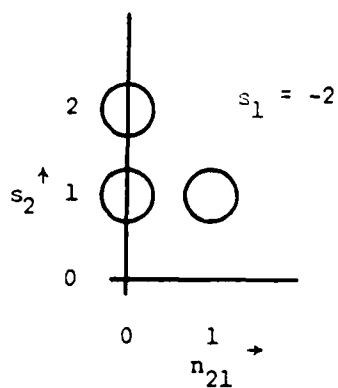
$$N_2 = 5$$

$$M_2 = 2$$

$$\lambda_{11} = \lambda_{21} = \lambda_{22} = .17$$

$$r_1 = r_2 = 1.$$

$$E(\text{Backorders}) = .1223$$

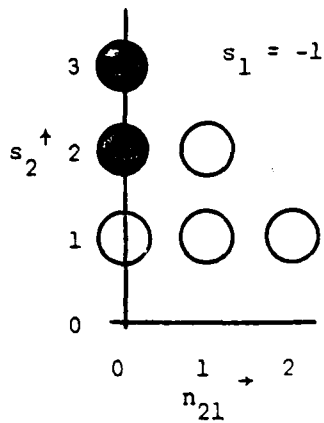


● : Substitute

○ : Don't Substitute

Example A. Optimal Policy

Figure 1.



$$N_1 = 4$$

$$M_1 = 2$$

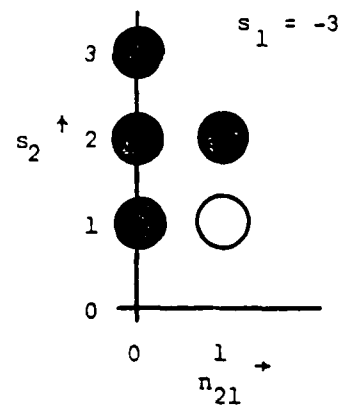
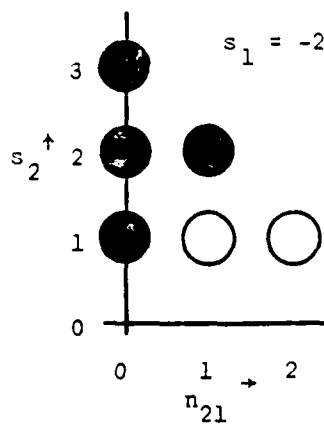
$$N_2 = 4$$

$$M_2 = 3$$

$$\lambda_{11} = \lambda_{21} = \lambda_{22} = .125$$

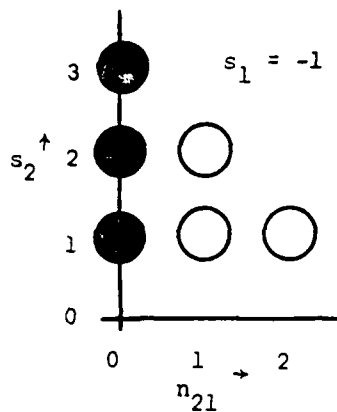
$$r_1 = r_2 = .25$$

$$E(\text{Backorders}) = .4558$$



Example B. Optimal Policy

Figure 2.



$$N_1 = 4$$

$$M_1 = 2$$

$$N_2 = 4$$

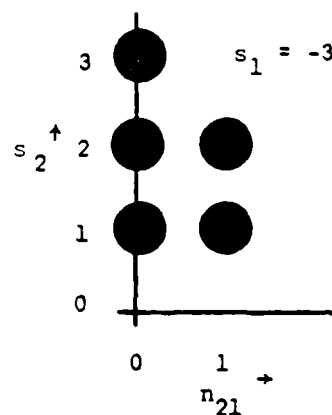
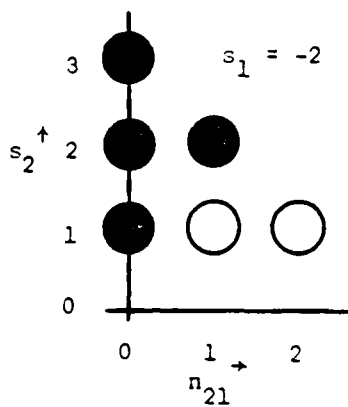
$$M_2 = 3$$

$$\lambda_{11} = .125$$

$$\lambda_{21} = \lambda_{22} = .1$$

$$r_1 = r_2 = .25$$

$$E(\text{Backorders}) = .3321$$



Example C. Optimal Policy

Figure 3.

IV. THE FORM OF AN OPTIMAL POLICY

To enable us to find approximations to an optimal policy, we examined the structure of the set of states, S , from which a type 2 item should be substituted into a type 1 unit. Clearly

$$S \subseteq \{(s_1, s_2, n_{21}) : s_1 < 0, s_2 > 0\}.$$

Every such subset gives rise to a control policy; however, there are three properties (monotonicity relationships) that one might expect the optimal subset (S_{opt}) to have. Willingness to perform a substitution depends on the number of type 1 units out of service, the number of type 2 items in spare stock, and the number of type 2 items already installed in type 1 units. Intuitively, for fixed s_2 and n_{21} , as s_1 decreases there is at least as great a need to substitute type 2 parts in the type 1 units. We express this as

$$MR1) (s_1, s_2, n_{21}) \in S_{opt}, s < s_1, \text{ implies } (s, s_2, n_{21}) \in S_{opt}.$$

Similarly, a greater supply of type 2 spares should imply an equal or greater willingness to substitute. This property is expressed as

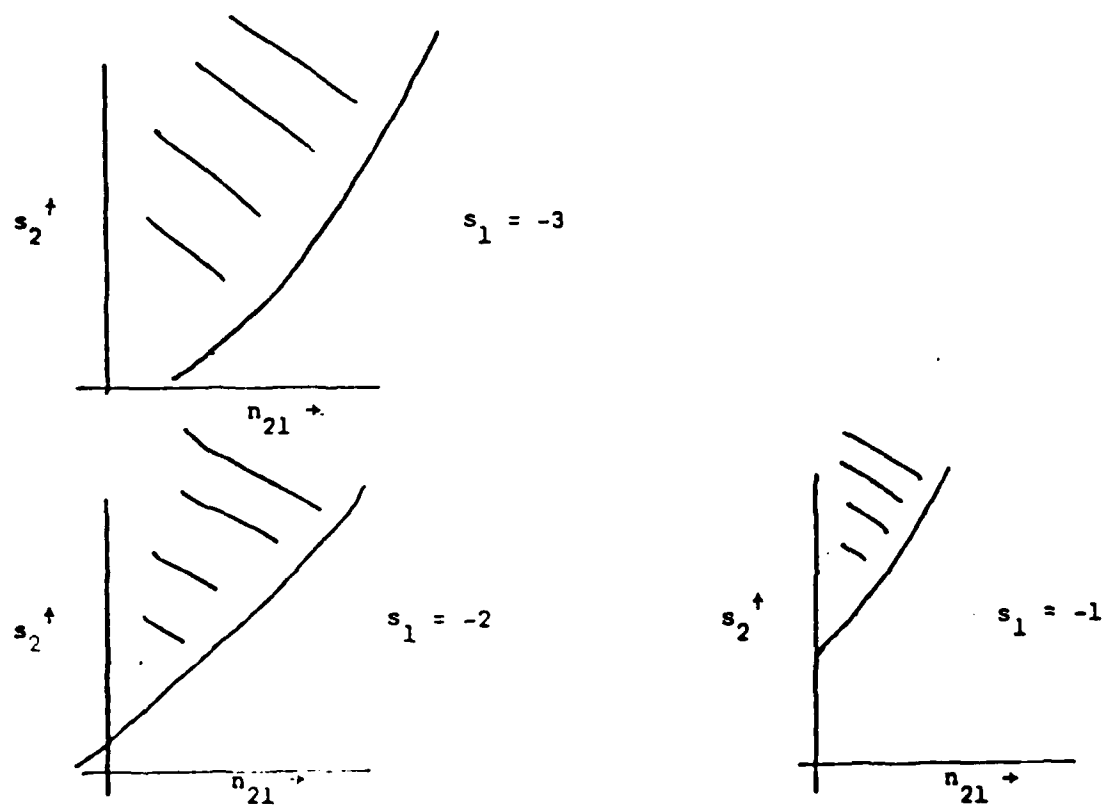
$$MR2) (s_1, s_2, n_{21}) \in S_{opt}, s > s_2, \text{ implies } (s_1, s, n_{21}) \in S_{opt}.$$

Finally,

$$MR3) (s_1, s_2, n_{21}) \in S_{opt}, n < n_{21}, \text{ implies } (s_1, s_2, n) \in S_{opt}.$$

Since S is a set in three-dimensional space we can draw its graph in sections; thus if we let each section correspond to a fixed value of s_1 we would expect the graph to look like that of Figure 4.

In each optimal strategy computed using the linear programming method described in Section III the properties MR1)-MR3) held. On the basis of this conjectured form of the optimal policy we developed a procedure to examine states for possible inclusion in S .



(The shaded region is the set of substituting states.)

Expected Form of an Optimal Policy

Figure 4.

V. APPROXIMATIONS TO AN OPTIMAL POLICY

To find a good approximation to the optimal substitution policy we employed a search technique which proceeds as follows: begin with S empty (i.e., with a policy which allows no substitution); repeatedly consider adding one point at a time to S by comparing the performance of the system using the augmented S with the current policy; then add the point if the level of performance is higher.

The first states considered are those for which $s_1 = -1$. In this set, the most likely candidate for membership in S is $(-1, M_2, 0)$: i.e., allow substitution when there is a type 1 unit out of service only if there are M_2 type 2 items in stock and none already installed in type 1 units. If this state is added to S , s_2 is successively decreased by one unit as long as the inclusion of $(-1, s_2, 0)$ improves the policy. Then n_{21} is incremented by 1 and $(-1, M_2-1, 1)$ is considered (note, if $n_{21} = 1$, then at most M_2-1 serviceable spare type 2 items can be in stock). Again, s_2 is decreased until there is no further policy improvement. We then increment n_{21} again. After all appropriate states of the form $(-1, s_2, n_{21})$ have been added to S , those for which $s_1 = -2$ are considered in a similar fashion. We take advantage of property MR1) to include automatically in S each state $(-2, s_2, n_{21})$ such that $(-1, s_2, n_{21})$ has already been added to S . This significantly reduces the number of comparisons which have to be made.

An alternate search pattern was considered in which $(-N_1, M_2, 0)$ is the first state examined for inclusion in S . In this state all type 1 units are out of service and all type 2 items are in spare stock. Intuitively, this is the state from which one would be most likely to allow substitution

and hence an appropriate starting point for the search. The disadvantage of this approach was that MPl) could not be used to increase the computational efficiency, and hence the first method was used.

The search technique involved being able to compare the performance of two policies. We developed two methods for this: one, analytic, which gave an exact number for the expected backorders under a given policy, and the second, simulation.

In the analytic method we observe that the substitution rules for a given policy and the transition probabilities depend only on the present state, hence the system is a Markov chain. The method consists of generating the steady state equations, finding the equilibrium probability distribution for the chain and calculating the expected number of backorders under the equilibrium distribution. This number can be compared to the expected backorders for the same policy augmented by one state as required by the search procedure.

A serious disadvantage of this method of policy comparison is related to the number of states in the chain and hence the number of equations to be solved for the equilibrium distribution. The number of states for a system with N_i units of type i and M_i items of type i is

$$(N_1 + M_1 + 1)(M_2 + 1)(M_2 + M_2/2 + 1) .$$

For example, a relatively small system with $N_1 = 10$ and $M_1 = 5$ has 1248 states. When the number of units and items grows by a factor of n , the number of states increases roughly by a multiple of n^3 . The time required to solve the equations corresponding to the enlarged system then increases by approximately a multiple of n^9 . We are, therefore, critically limited in the size of the system we can investigate using the Markov analysis to compare policies.

The major advantage of this approach is that the answers are exact and hence two policies can be compared. The similar results obtained by linear programming and by the search technique with analytic policy comparison suggested that the search mechanism is valid and encouraged the development of a more efficient method of policy comparison.

The simulation method, which provides the only tool suitable for the analysis of large systems, performs two simulations, one for each policy and uses the state at which the policies differ as a starting point for the simulation. In one, the substitution of a type 2 part is made, and in the other it is not made. The two simulations are then run until they both reach the same state. This state is not necessarily the one in which the simulations started. If either system reaches the initial state in which the desirability of substitution is being questioned, the substitution is not made.

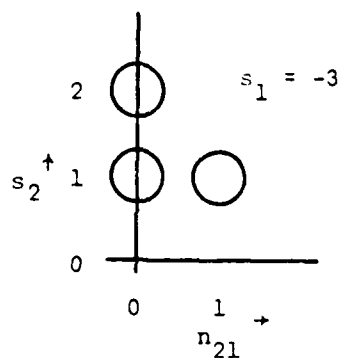
When the two systems reach the same state the run is terminated and the difference between the number of backorder-days is recorded; call the result (say the number of backorder-days with the extra substitution minus the backorder-days without it) for run i B_i .

We wish to determine $E(B_1)$, since if $E(B_1) > 0$ we should not perform the additional substitution while, if $E(B_1) < 0$ we should. To estimate $E(B_1)$ we computed $\sum_{i=1}^n B_i$ and $\sum_{i=1}^n B_i^2$; we could then estimate $E(B_1)$ and $V(B_1)$ and construct confidence intervals for $E(B_1)$. Large groups of runs were made; after each group of runs a confidence interval was computed at a selected confidence level. If this confidence interval did not include the origin the procedure was terminated and the appropriate strategy was selected as optimal. Moreover, if the confidence interval did include zero but was shorter than some pre-selected tolerance level, the procedure was terminated and it was concluded that both policies gave nearly the same performance level.

(Actually in our runs this outcome did not occur.) It should be noted that although we were using a sequential procedure we used analysis appropriate for a single sample and that this is not precisely correct. However, in each case the results obtained by simulation were the same as those obtained by analytic comparison. Moreover, the simulation was computationally superior in two respects: the amount of computer time did not grow as rapidly with increased system size as it did for the analytic method, and, perhaps even more relevant in terms of absolute limitations, the simulations did not require the large amounts of storage necessitated by solving such large systems of equations. Substitution sets (S) generated by the search procedure for the same examples presented in Section III are found in Figures 5, 6, and 7.

In conclusion, we make the following observations:

- 1) the exact solutions obtained through linear programming and the approximate solutions obtained with the search support the assumption that the form of the optimal policy satisfies MR1) - MR3);
- 2) comparison of exact results and those obtained by a search with analytic policy evaluation indicate that a search is an effective method of policy improvement;
- 3) the correlation between results of search with analytic comparison and simulation comparison indicates that the simulation method is valid and hence gives us a method for finding an approximation to an optimal policy which can be applied to large systems.



$$N_1 = 5$$

$$M_1 = 2$$

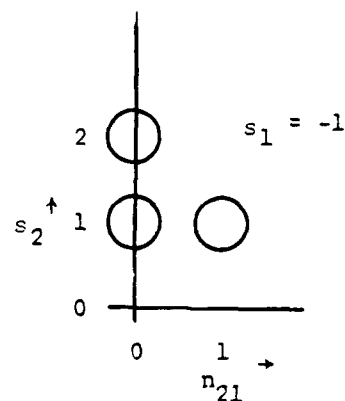
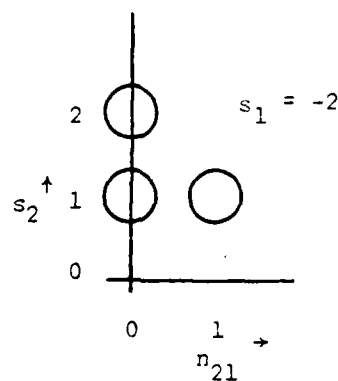
$$N_2 = 5$$

$$M_2 = 2$$

$$\lambda_{11} = \lambda_{21} = \lambda_{22} = .17$$

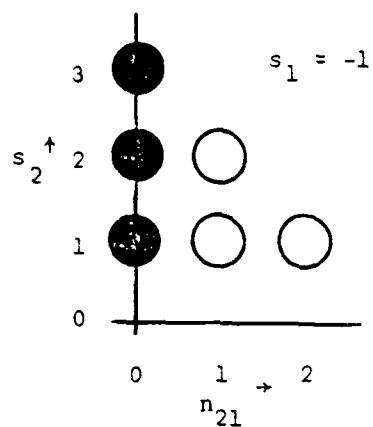
$$r_1 = r_2 = 1.$$

$$E(\text{Backorders}) = .1226$$



Example A. Policy Obtained from Search

Figure 5.



$$N_1 = 4$$

$$M_1 = 2$$

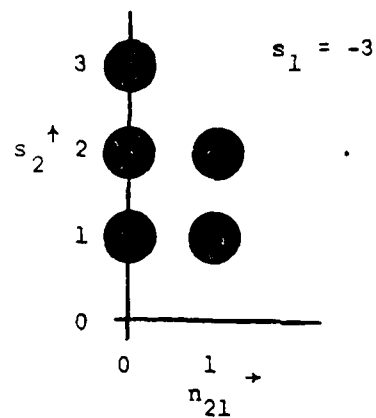
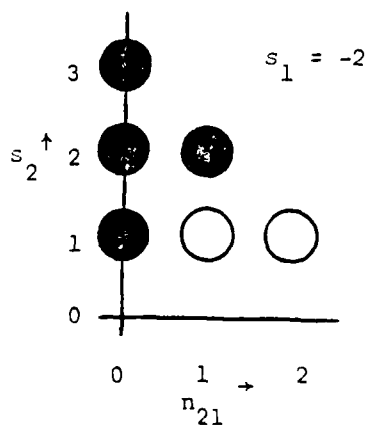
$$N_2 = 4$$

$$M_2 = 3$$

$$\lambda_{11} = \lambda_{21} = \lambda_{22} = .125$$

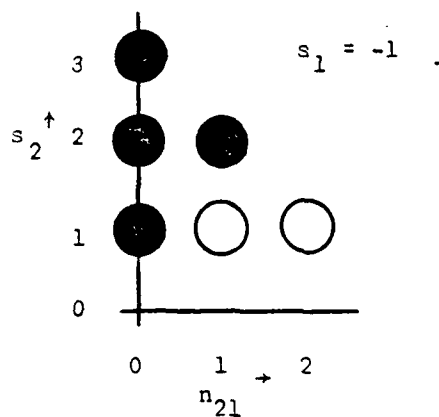
$$r_1 = r_2 = .25$$

$$E(\text{Backorders}) = .4585$$



Example B. Policy Obtained from Search

Figure 6.



$$N_1 = 4$$

$$M_1 = 2$$

$$N_2 = 4$$

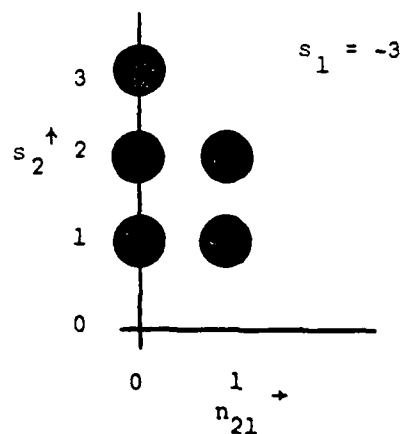
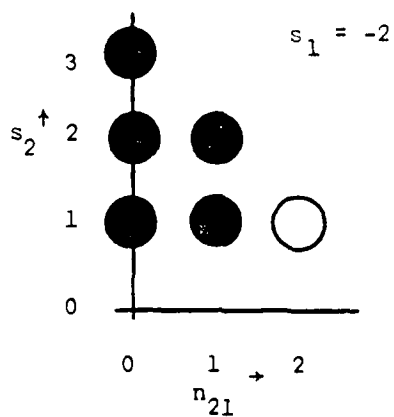
$$M_2 = 3$$

$$\lambda_{11} = .125$$

$$\lambda_{21} = \lambda_{22} = .1$$

$$r_1 = r_2 = .25$$

$$E(\text{Backorders}) = .3351$$



Example C. Policy Obtained from Search

Figure 7.

VI. A SUMMARY AND COMMENTS CONCERNING FUTURE EFFORTS

While we now have a method for policy approximation which can be used on large systems, there are several possibilities for further work with this model. Continued investigation of the structure of an optimal policy (i.e., is it linear in any of its variables?) might suggest a reduction in the number of states to be considered for inclusion in S during a search for an approximation to the optimal policy. A more precise comparison of the performance of the exact and approximate solutions should be made to find a balance between reduced backorders and computational accessibility.

In addition, we plan to use the insights obtained through the study of this highly simplified inventory model as a basis for our future efforts to study more complex situations.

Unfortunately, many real-world considerations are not addressed in our simplified model. For example, we have ignored the fact that a) sometimes a family of substitutable items may consist of more than two items, b) items are normally stocked at more than one location, c) the failure and repair distributions may not be stationary, d) the planning horizon may be of such a short duration that infinite horizon models may be inappropriate, and e) the system stock level for each item may not remain constant for an extended period of time. We plan to address many of these issues in our future work.

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20. for this problem there exist optimal stationary Markov control policies. Next we formulate a continuous time model and show how to find the optimal stationary Markov control policy using linear programming. Unfortunately, this approach is impractical for solving most real problems. Consequently we have established and explored some of the properties that we feel an optimal policy should possess. A discussion of these properties is given in Section IV. Lastly, we will describe a heuristic that can be used to find a good policy. This method is an efficient simulation search method that finds policies having the properties we conjecture an optimal policy should possess.

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Computational Considerations for a Single Location
Inventory Problem for two Interchangeable Recoverable Items

by

Carol C. Shilepsky

Table of Contents

I. Introduction	1
II. Formulation of the model and its solution by linear programming	3
III. Providing an input basis for the linear programming solution	8
IV. Restricting the state space for the linear programming solution	9
V. Summary and Comments Concerning Future Work	17
VI. Bibliography	18

1. Introduction

The effective management of recoverable spare stock leads to many important problems in inventory theory (See [1] for references to earlier work). Very little work has been done in the case that recoverable items are interchangeable or that one can be substituted for another during repair. Such a problem can arise when one part is more useful than another and can be used to satisfy a demand for itself or for another type part, while the less useful type part can only be used to satisfy a demand for itself. A substitution (i.e., of the more useful part for the less useful part) may lead to subsequent shortages of the more useful type part, and, although it will prevent an immediate backorder, may not always be the best action. An inventory policy would specify conditions under which a substitution should be made.

In [1] Heath, Muckstadt and Shilepsky consider the following interchangeability/substitutability problem for two items that fail at a single location. The failure processes for each type of item are independent, stationary Poisson processes. Failed items are repaired and the repair times are exponentially distributed. Furthermore, the system is closed, that is, no items are added to or removed from the system.

They first consider a discrete time formulation of this model and show that it is a Markovian decision problem for which there exist optimal stationary Markov control policies. They next formulate a continuous time model and show how to find the optimal stationary control policy using linear programming. Because of the size of most real problems, the linear programming approach cannot be applied; however, its application to small

problems led to conjectures concerning properties that an optimal solution should possess [Heath, Muckstadt and Shilepsky]. Based on these properties an efficient simulation search method was developed to find good, but not optimal, policies.

In this paper we discuss computational considerations relating to the linear programming solution of small problems. We first examine a technique for improving the efficiency of the linear programming solution by providing a good starting basis. Second we discuss techniques for reducing the number of variables in the linear programming problem. These techniques enable us to solve problems which had previously been inaccessible either because of their size or numerical problems.

In [2] Heath, Muckstadt and Shilepsky present a scaling technique by which one may approximate a large problem by a small one, find the optimal solution for the latter and use it to find a solution to the large problem. The effectiveness of the scaling technique depends heavily on the ability to solve small problems efficiently and accurately.

II. Formulation of the Model and Solution by Linear Programming

We consider the following model. We suppose that there are two types of items in the system: type 1 and type 2. The assemblies in which these items are installed are called units; we also assume that there are two types of units and each unit contains only one item of the types considered. All items are assumed to be stored at a single location.

We assume that items of type 2 can be placed in units of type 1 or type 2, but that items of type 1 can be placed only in units of type 1. Finally, we assume that times to failure and repair times are exponentially distributed.

For the model we suppose that:

- 1) There are N_i units of type i and $N_i + M_i$ items of type i are available (i.e., there are M_i spare items of type i),
- 2) The state of the system at any instant can be specified by the five numbers:

n_{ij} , the number of type i items installed in type j units,
 $i \geq j$.

m_i , the number of serviceable type i units in spare stock.

The number of type i items in repair is then given by $N_i + M_i - (m_i + \sum_{j=1}^i n_{ij})$ and the number of back-orders associated with type j units is $N_j - \sum_{i=j}^2 n_{ij}$. We suppose that the failure times of items of type i installed in units of type j are independent exponential random variables with mean $1/\lambda_{ij}$, and that repair times are independent, exponential random variables with mean $1/r_i$. The measure of performance of the system under any substitution policy is the expected number of backorders when the system is in equilibrium.

Under any reasonable control policy (i.e., rule specifying which items to use to repair various units under all possible circumstances) the number of variables necessary to describe the state of the system can be reduced from the five above to three as follows: it would clearly be unreasonable to allow backorders for type i units if there were serviceable spare stock on hand; thus, $N_i - \sum_{j=1}^2 n_{ij}$ and m_i cannot both be positive. We thus set $s_i = m_i - [N_i - \sum_{j=1}^2 n_{ij}]$, which is the net inventory of items of type i (it is negative if there are backorders associated with type i units). The variables s_1, s_2 and n_{21} then describe the state of the system.

Clearly all reasonable strategies will use items of type i to satisfy demands for units of type i whenever these items are available. Thus the only question which needs to be considered is under what conditions to allow the use (i.e., substitution) of a type 2 item in a type 1 unit. Moreover, in this situation there always is an optimal stationary Markov policy (see Ross [2]); and hence it suffices to identify the set of states (called S , or the substituting states) in which a type 2 item would be installed in a type 1 unit. Clearly

$$S \subseteq \{(s_1, s_2, n_{21}) : s_1 < 0, s_2 > 0\}$$

The set of states corresponding to the policy which minimizes the expected number of backorders can be found by solving a linear programming problem in which the variables are non-substituting states: one corresponding to each possible state of the system, plus substituting states: one corresponding to each state from which a substitution could be made ($s_1 < 0$ and

$s_2 > 0$). In the former, no substitution is made; in the latter, a substitution is assumed. There is a constraint corresponding to each non-substituting state and the objective function is the expected number of backorders corresponding to the states in the solution. The basic variables in the final solution are those states in the optimal policy. See [1], Heath, Muckstadt, and Shilepsky for further discussion of the linear programming formulation.

A program was written to generate the input for a linear programming package, MPSX. The output was then submitted to MPSX. We present the results from two problems in Example 1 and Example 2. Note that since at each state, an optimal policy merely specifies "substitute" or "don't substitute", it suffices to graph the set of states at which one would perform a substitution.

Example 1

$$N_1 = 4$$

$$M_1 = 2$$

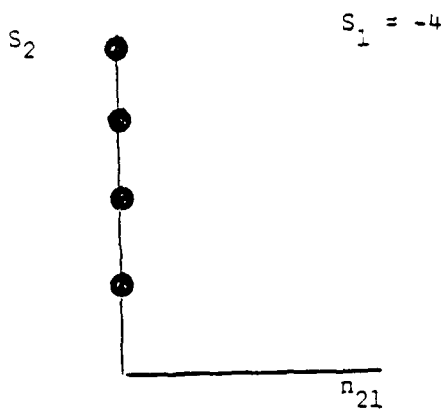
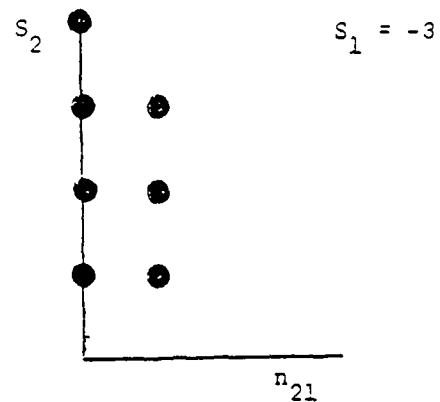
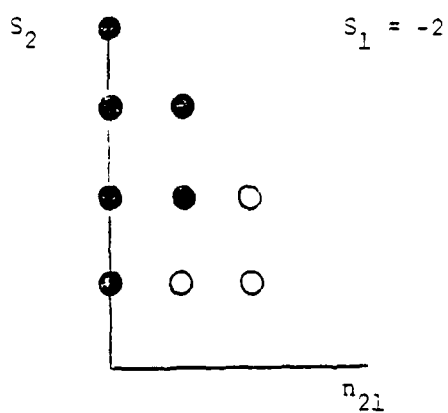
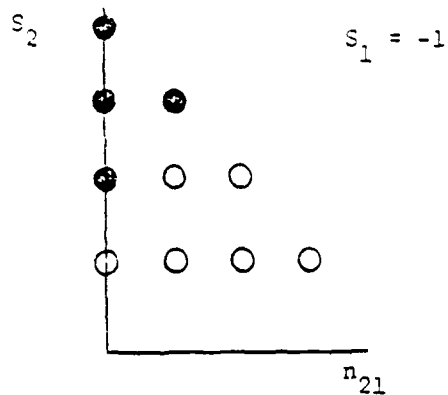
$$N_2 = 4$$

$$M_2 = 4$$

$$\lambda_{11} = .5$$

$$\lambda_{12} = \lambda_{22} = 1$$

$$r_1 = r_2 = 1$$



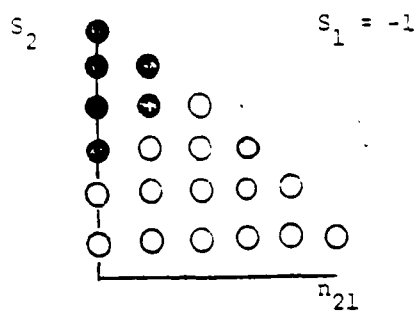
● substitute

○ do not substitute

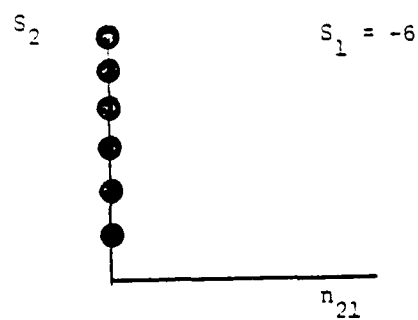
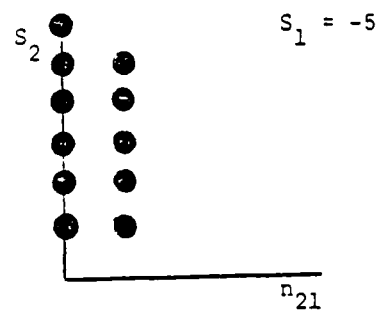
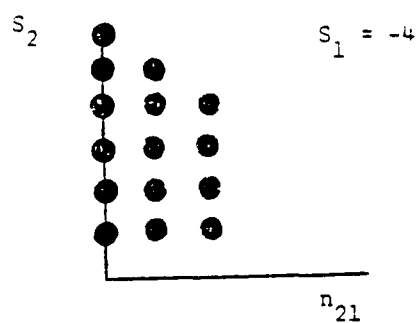
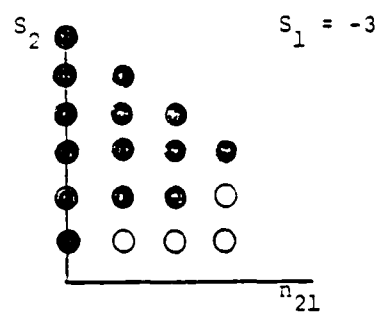
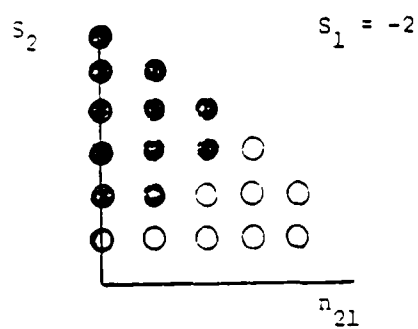
Figure 1.

Example 2

7



$$\begin{aligned} N_1 &= 6 \\ M_1 &= 3 \\ N_2 &= 6 \\ M_2 &= 6 \\ \lambda_{11} &= .5 \\ \lambda_{12} &= \lambda_{22} = 1 \\ r_1 &= r_2 = 1 \end{aligned}$$



● substitute
○ do not substitute

Figure 2.

III. Providing an Input Basis for the Linear Programming Solution

A modification of the linear programming method for finding the optimal solution was made to allow specification of a starting basis for MPSX. Most of the computational time for determining the optimal policy appeared to be used finding an initial feasible solution. Any policy will give a feasible solution and one possible policy is to allow no substitution. Hence the basis of vectors which contains only non-substituting states should correspond to a feasible solution.

The program which generated input for MPSX was expanded to produce a second file, specifying the variables corresponding to each of the non-substituting states as a basic variable in the starting solution. In a small problem the number of iterations was reduced by this technique from 53 to 5. Larger problems which had terminated because of accumulated roundoff errors were successfully solved.

Larger problems such as that of Example 2 were difficult to solve even with the starting basis. For such problems a further refinement was successfully used. An approximation of the optimal solution was obtained by solving a smaller problem and the scaling to be discussed in later work. The approximation was then used as a starting basis for MPSX. This reduced the number of iterations and again allowed us to solve problems for which the linear programming algorithm had terminated because of numerical difficulties.

IV. Restricting the State Space for the Linear Programming Solution

The number of states in a system with N_i units of type i and M_i spares of type i is

$$(N_1 + M_1)(N_2 + \frac{M_2}{2} + 1)(M_2 + 1).$$

Thus as the values of N_i and M_i grow, the number of states for the control problems grows very rapidly. We therefore considered two methods for reducing the number of states allowed:

1. Restriction of the set of policies considered.
2. Removal of a class of "very unlikely" states.

Such restrictions decrease the number of variables for the linear programming problem and hence increase the size of the problems we can consider. These reductions also lead to greater efficiency in generating the input for MPSX.

1. Restriction of the set of policies: One component of the state, namely n_{21} , is under our direct control. If we choose to, we can control the process in such a way that n_{21} never gets very large. If we restrict consideration to those policies which maintain $n_{21} \leq n_{21} \text{ MAX}$ we can reduce the states allowed to those having $n_{21} \leq n_{21} \text{ MAX}$. The solution then found is still an optimal solution but to a smaller problem, one in which certain types of substitutions are prohibited. A reasonable but possibly not optimal solution for the unrestricted problem can then be obtained for each s : by an extrapolation of the restricted solution. A typical restriction is $n_{21} \leq 2$ or $n_{21} \leq 3$.

A comparison of solutions for restricted and unrestricted state spaces suggests that very little accuracy is lost by the restriction. This is partially due to the fact that states with large n_{21} are reached so infrequently that their omission changes the behavior of the system very little.

Example 1 in Section II with $N_1 = 4$, $M_1 = 2$, $N_2 = 4$, $M_2 = 4$ was set up and solved using only states with $n_{21} \leq 2$. The number of states was reduced from 522 to 361 and the solution shown in Figure 3 was obtained.

The restriction $n_{21} \leq 2$ in Example 2, Section II with $N_1 = 6$, $M_1 = 3$, $N_2 = 6$, $M_2 = 6$ reduced the number of states from 791 to 421 and gave the solution shown in Figure 4.

Note in Figures 2 and 3 that because $n_{21} \leq 2$, no state with $n_{21} > 1$ will be in the set of substituting states, but that the results obtained for $n_{21} = 0$ or 1 agree exactly with those obtained for the full space. We can use these results to guess (by, say, linear extrapolation to larger values of n_{21}) good policies for the original problem, or, alternatively, we can carry out the computation with $n_{21} \leq 3$ using the results obtained to choose a good initial basis.

2. Removal of a class of "very unlikely" states: Although the number of possible states grows very rapidly, the number of states which "can reasonably occur" grows much more slowly. For example, if no substitution is allowed, standard approximations show that the number of type 1 items in repair is approximately normally distributed with mean $\frac{N_1 \lambda_{11}}{\mu_{11}}$ and variance $\frac{N_1 \lambda_{11}}{\mu_{11}}$. (The corresponding result holds for type 2 items.) If we let r_1 (r_2) be the number of type 1 (2) items in repair, we see that

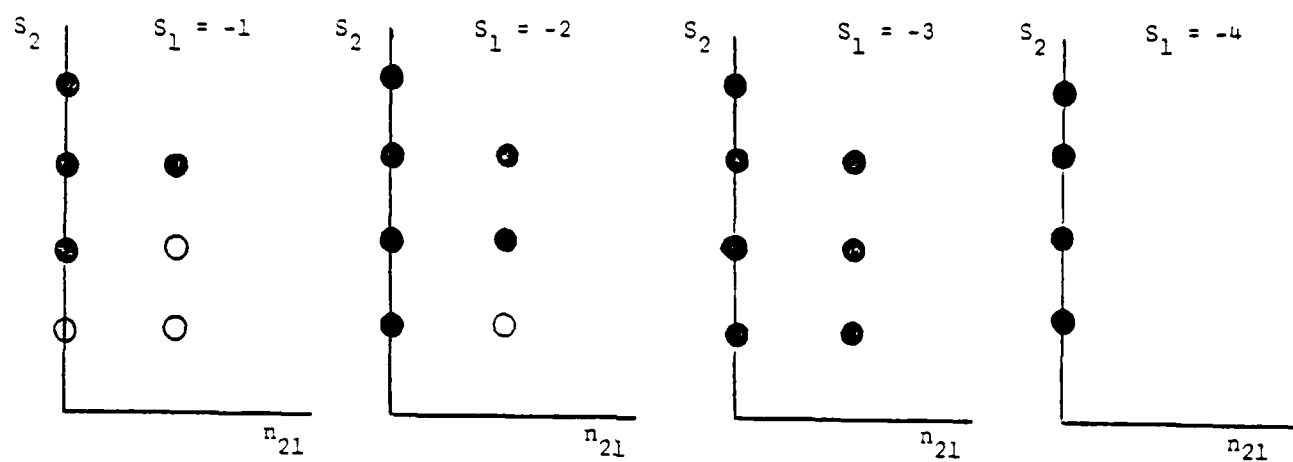


Figure 3.

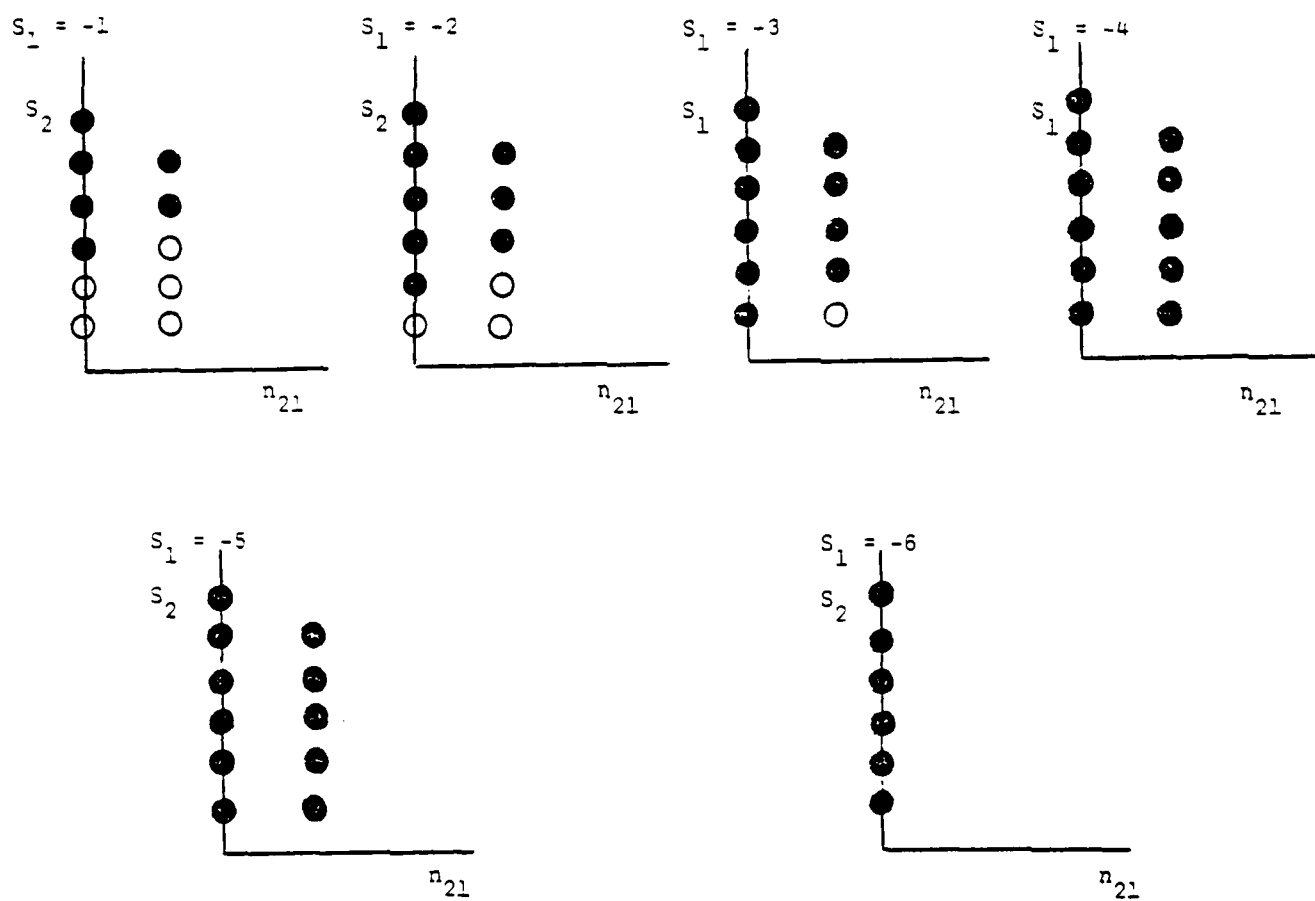


Figure 4.

$$T = \left(\frac{r_1 - \frac{N_1 \lambda_{11}}{\mu_1}}{\sqrt{\frac{N_1 \lambda_{11}}{\mu_{11}}}} \right)^2 + \left(\frac{r_2 - \frac{N_2 \lambda_{22}}{\mu_{22}}}{\sqrt{\frac{N_2 \lambda_{22}}{\mu_{22}}}} \right)^2$$

is approximately a χ^2 random variable with 2 degrees of freedom.

It can be seen (most easily by considering scaling results to be presented in a later work) that allowing some substitution does not change the distribution of T very much.

We thus use T to identify a class of states to be eliminated...we rule out all states for which T is greater than some tolerance limit TOL. In doing this we do not allow those transitions which would lead to an eliminated state. All other transition probabilities remain the same.

In Example 1, for example, eliminating those states for which $T > 3$ reduces the number of states from 522 to 287. The policy obtained remains exactly the same. If all states having $T > 2.5$ are eliminated, the number of states is further reduced to 211. The results for this case are shown in Figure 5.

Since some states were eliminated, the policy is not defined at some of the original states. Where it is defined it agrees with the previously obtained policy except at the point indicated by \square in Figure 5.

As a further illustration, Example 2 with all states having $T > 3$ eliminated yielded the policy shown in Figure 6.

In this case the number of states was reduced from 791 to 496. Again the solution agrees with the exact solution for those states not eliminated.

A reasonable approach to restricting the statespace seems to be a combination of the two techniques discussed. The following results obtained

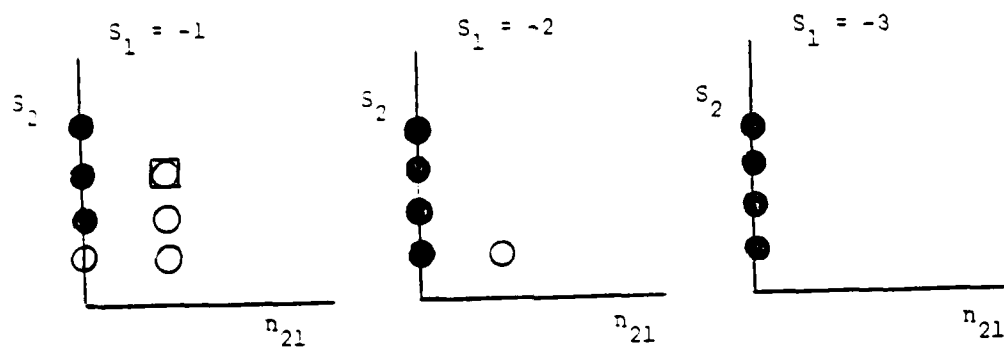


Figure 5.

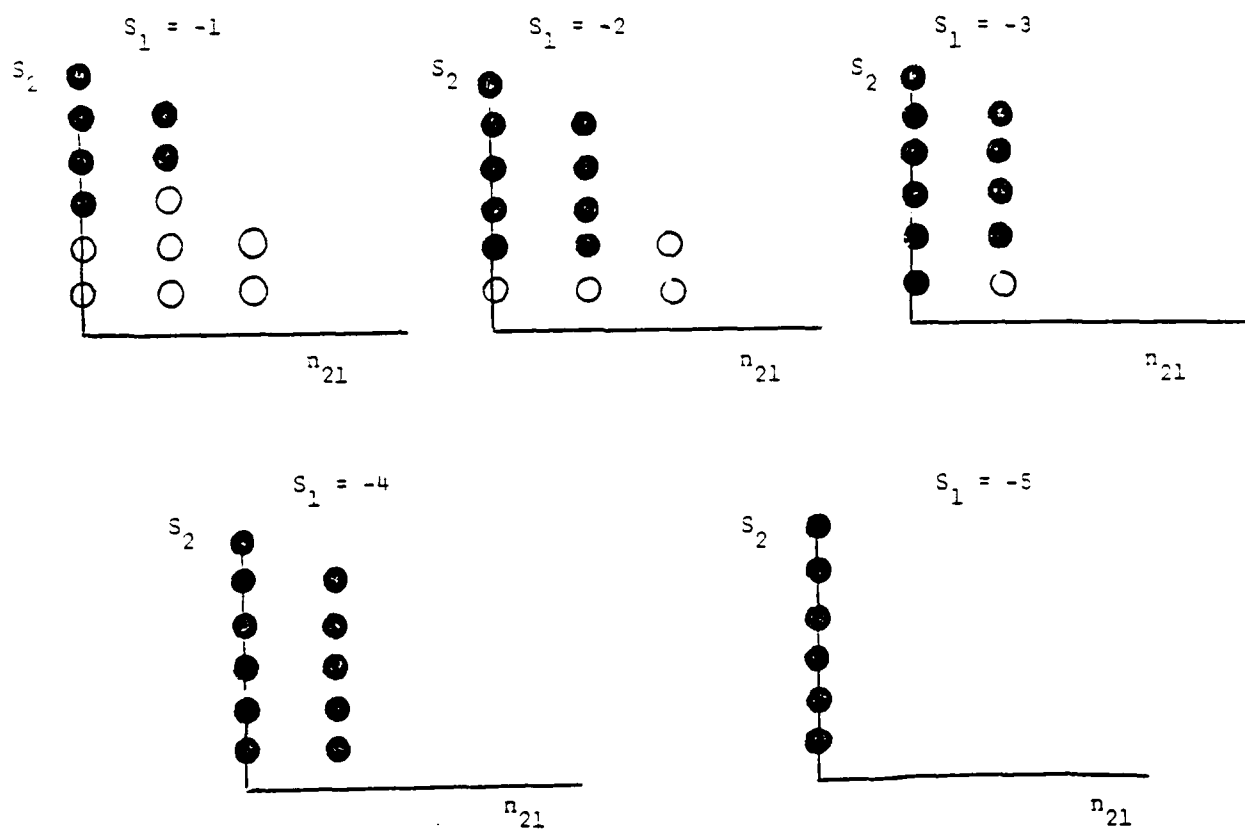


Figure 6.

for Example 1 with $n_{21} \leq 2$ and TOL 3 and only 180 states agreed with the exact results. A more useful result is obtained here with only 180 states then in Figure 5 with 211 states indicating that judicious combination of the two techniques is appropriate.

Example 2 in Section II with $N_1 = 6$ $M_1 = 3$ $N_2 = 6$ $M_2 = 6$ was run with $n_{21} \leq 2$ and TOL 3. Here the number of states was reduced from 791 to 332 obtaining, again, results that agree with the exact solution.

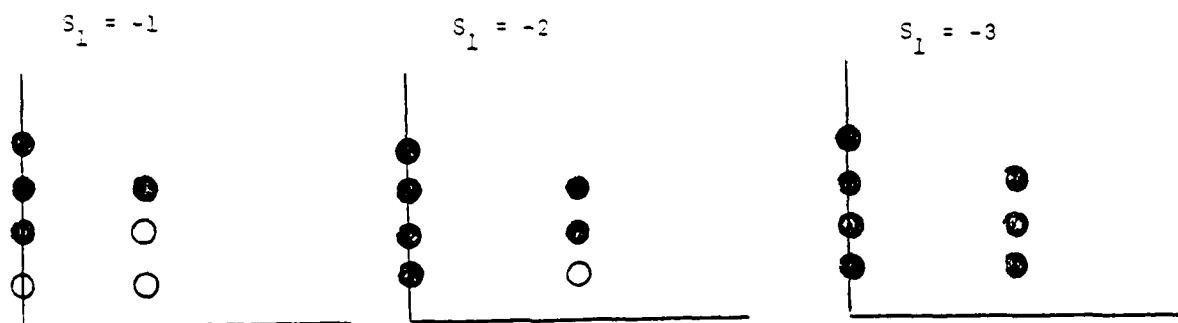


Figure 7.

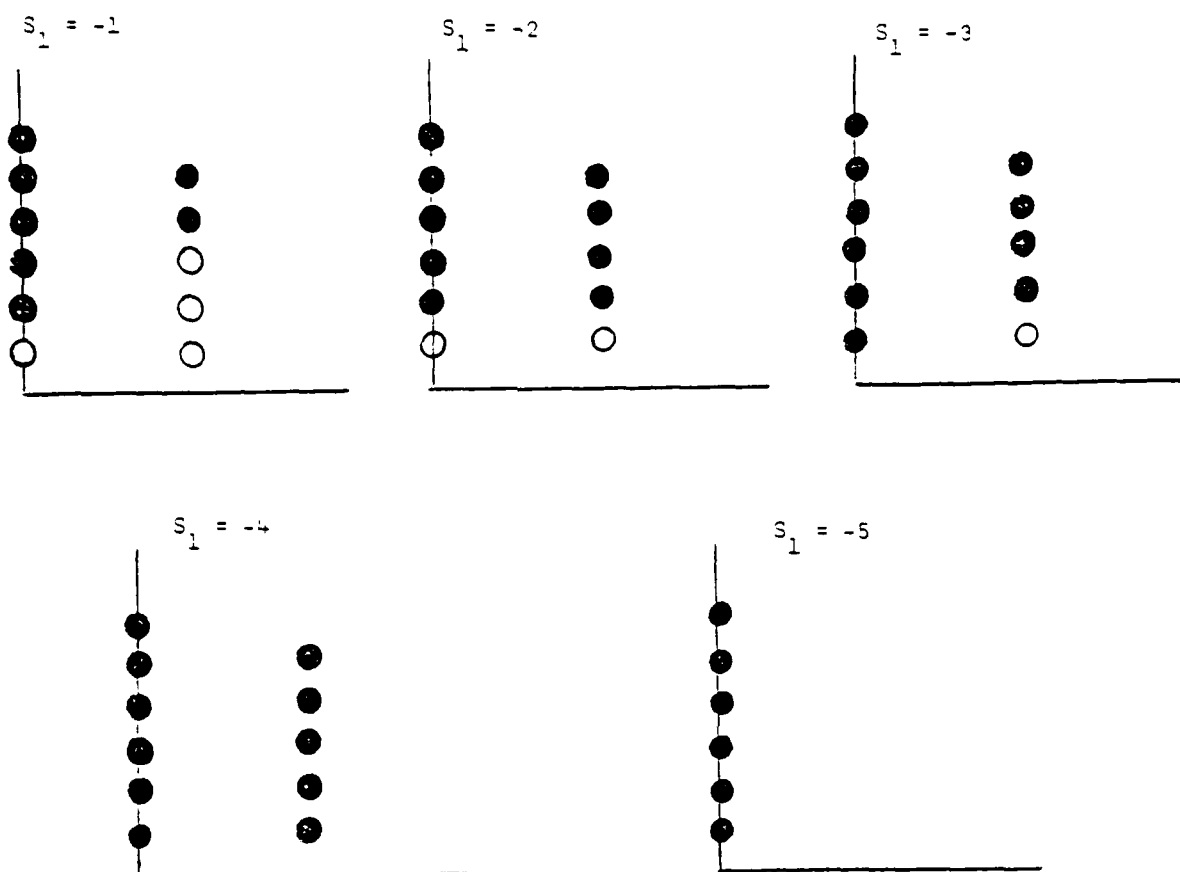


Figure 8.

V. Summary and Comments Concerning Future Work

We have now improved our ability to find optimal or near-optimal solutions using linear programming on small examples. By providing a starting basis for MPSX and by eliminating states from those we consider we have reduced the size of the problem and the amount of time MPSX takes to solve it. This has been helpful in reducing the cost of each solution and in lessening the effect of roundoff errors. We have been able to obtain optimal or near-optimal solutions for many cases which had been previously unsolvable.

The effects of this have been two-fold. First, we have attained greater insight into the structure of the set of substituting states in an optimal solution. For example, the monotonicity properties proposed in [1] have held in each case we investigated. Second, we now have the ability to solve sufficiently large problems that the scaling technique to be discussed in future work will be an effective way to solve problems of any size.

We plan to extend this approach to finding optimal policies for finite state Markovian decision problems.

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SIMULATION TECHNIQUES FOR AN INVENTORY
SYSTEM WITH INTERCHANGEABLE RECOVERABLE ITEMS

by

Carol C. Shilepsky

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Table of Contents

I. Introduction	1
II. Formulation of the Model	1
III. Search for Optimal Policies	3
IV. Policy Comparison by Simulation and the Use of Variance Reducing Techniques	5
V. Summary and Comments Concerning Future Work	9
VI. Bibliography	10

I. Introduction

In [1] Heath, Muckstadt and Shilepsky consider an inventory model for recoverable items in which one item may be substituted for another during repair. The model is formulated as a Markovian decision problem for which there exists an optimal stationary control policy. The optimal policy can be found by linear programming, but the size of most real problems makes this approach impractical. Its application to small problems led to conjectured properties that an optimal policy should satisfy. Based on these properties, an efficient search method was developed to find good, but not necessarily optimal policies. The search involves a comparison of policies by analytic methods or simulation. This paper focuses on policy comparison by simulation and several variance reducing techniques that contribute to the efficiency of the simulation.

II. Formulation of the Model

We consider the following model. We suppose that there are two types of items in the system: type 1 and type 2. The assemblies in which these items are installed are called units; we also assume that there are two types of units and each unit contains only one item of the types considered. All items are assumed to be stored at a single location.

We assume that items of type 2 can be placed in units of type 1 or type 2, but that items of type 1 can be placed only in units of type 1. Finally, we assume that times to failure and repair times are exponentially distributed.

For the model we suppose that:

- 1) There are N_i units of type i and $N_i + M_i$ items of type i are available (i.e., there are M_i spare items of type i),
- 2) The state of the system at any instant can be specified by the five numbers:

n_{ij} , the number of type i items installed in type j units, $i \geq j$.

m_i , the number of serviceable type i units in spare stock.

The number of type i items in repair is then given by $N_i + M_i - (m_i + \sum_{j=1}^i n_{ij})$ and the number of back-orders associated with type j units is $N_j - \sum_{i=j}^2 n_{ij}$. We suppose that the failure times of type i installed in units of type j are independent exponential random variables with mean $1/\lambda_{ij}$, and that repair times are independent, exponential random variables with mean $1/r_i$. The measure of performance of the system under any substitution policy is the expected number of backorders when the system is in equilibrium.

Under any reasonable control policy (i.e., rule specifying which items to use to repair various units under all possible circumstances) the number of variables necessary to describe the state of the system can be reduced from the five above to three as follows: it would clearly be unreasonable to allow backorders for type i units if there were serviceable spare stock on hand; thus, $N_i - \sum_{j=1}^2 n_{ij}$ and m_i cannot both be positive. We thus set $s_i = m_i - [N_i - \sum_{j=1}^2 n_{ij}]$, which is the net inventory of items of type i (it is negative if there are

backorders associated with type i units). The variables s_1, s_2 and n_{21} then describe the state of the system.

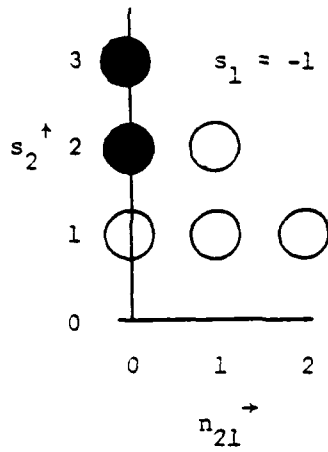
Clearly all reasonable strategies will use items of type i to satisfy demands for units of type i whenever these items are available. Thus the only question which needs to be considered is under what conditions to allow the use (i.e., substitution) of a type 2 item in a type 1 unit. Moreover, in this situation there always is an optimal stationary Markov policy (see [1]); and hence it suffices to identify the set of states (called S , or the substituting states) in which a type 2 item would be installed in a type 1 unit.

The set of states corresponding to the policy which minimizes the expected number of backorders can be found by solving a linear programming problem. Since at each state an optimal policy merely specifies "substitute" or "don't substitute", it suffices to graph the set of states at which one would perform a substitution. We present the solution for a typical problem in Figure 1.

The most serious drawback to the linear programming solutions is related to the number of states in the Markov chain and hence the number of constraints in the linear programming formulation. See [1], p. 10 for discussion. We therefore investigated alternative approaches.

III. Search for Optimal Policies

To find a good approximation to the optimal policy we employ a search technique which begins with the set of substituting states, S , empty and repeatedly considers adding states one at a time to S by



$$N_1 = 4$$

$$M_1 = 2$$

$$N_2 = 4$$

$$M_2 = 3$$

$$\lambda_{11} = \lambda_{21} = \lambda_{22} = .125$$

$$r_1 = r_2 = .25$$

$$E(\text{Backorders}) = .4558$$

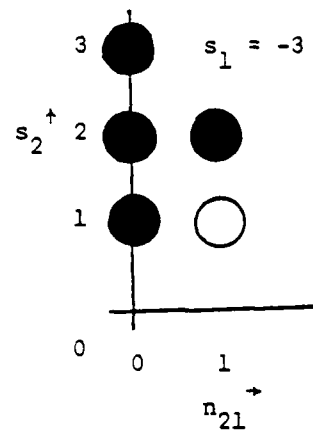
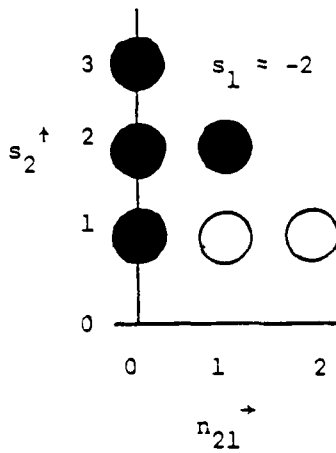


Figure 1.

comparing the performance of the system using the augmented set of substituting states with the performance using the current policy. We developed two methods for policy comparison: one, analytic which gives an exact number for the expected backorders under a given policy, and the second, simulation. Similar results obtained by linear programming and the search with analytic policy comparison suggest that the search mechanism is valid. However, the search with analytic policy comparison requires roughly the same amount of computer time and storage as solution by linear programming. Hence simulation was investigated for more efficient policy comparison. In each case examined, the results obtained by analytic and simulation comparison were the same. See [1, p. 10] for a discussion of the search technique and comparative results.

IV. Policy Comparison by Simulation and the Use of Variance Reducing Techniques

The simulation method performs two simulations, one for each policy and uses the state at which the policies differ as a starting point for the simulation. In one, the substitution of a type 2 part is made, and in the other it is not made. The two simulations are then run until they both reach the same state. This state is not necessarily the one in which the simulations started. If either system reaches the initial state in which the desirability of substitution is being questioned, the substitution is not made.

When the two systems reach the same state the run is terminated. This is repeated a large number of times, and the difference in backorder days for the system with substitution and the system without substitution is recorded. We compute the sample mean for the difference in backorder days, \bar{X} , and sample variance, σ^2 . We wish to determine $E(\bar{X})$ since if $E(\bar{X}) < 0$, the substitution should be made and if $E(\bar{X}) > 0$, it should not. A confidence interval is constructed at a selected confidence level; if this confidence interval does not include the origin, the procedure is terminated and the appropriate strategy selected as optimal. If the confidence interval does include the origin, the results of another large group of runs are included with the previous results. The procedure is repeated until the confidence interval does not include the origin or is shorter than some pre-selected tolerance level. In the latter case the decision to substitute or not is based on the sign of the sample mean, but no level of confidence assured. In these cases both policies give nearly the same performance level and hence an incorrect choice would have little effect on system behavior.

The number of runs in each group is large enough so that the central limit theorem may be invoked and confidence intervals constructed. It should be noted that we are conducting a sequential test, but in the construction of confidence intervals, applying non-sequential techniques. In practice, however, if the number of runs in the first group is large enough, the results are definitive and a second group unnecessary.

A major contribution to variance reduction of \bar{X} comes through the parallel simulation of the systems with and without substitution. We

note that if $\bar{X} = \bar{X}^n - \bar{X}^s$ where \bar{X}^n and \bar{X}^s are sample means for systems without and with substitution then

$$\text{Var}(\bar{X}) = \text{Var}(\bar{X}^n - \bar{X}^s) = \text{Var}(\bar{X}^n) + \text{Var}(\bar{X}^s) - 2 \text{Cov}(\bar{X}^n, \bar{X}^s).$$

Failure and repair times for parts which are common to each system are the same, inducing a high correlation between \bar{X}^n and \bar{X}^s .

The search with policy comparison by simulation was run for the problem in Figure 1 keeping track of $\text{Var}(\bar{X}^n)$, $\text{Var}(\bar{X}^s)$ and $\text{Cov}(\bar{X}^n, \bar{X}^s)$ after each decision. We present these results in Figure 2, also including the variance if the systems were uncorrelated, $\text{Var}(\bar{X}^n) + \text{Var}(\bar{X}^s)$, and the variance with the induced correlation, $\text{Var}(\bar{X}^n) + \text{Var}(\bar{X}^s) - 2 \text{Cov}(\bar{X}^n, \bar{X}^s)$ in the last two columns.

We note that \bar{X}^n and \bar{X}^s are highly correlated and that the average ratio of correlated to uncorrelated variances is .1358, hence we have decreased the number of runs necessary to obtain the same level of confidence by a factor of approximately 8.

$\text{Var}(\bar{X}^n)$	$\text{Var}(\bar{X}^s)$	$\text{Cov}(\bar{X}^n, \bar{X}^s)$	Variance if uncorrelated	Variance with correlation
68.98	59.5	53.17	128.48	22.14
37.06	55.61	58.56	142.67	25.55
44.25	36.13	31.47	80.38	17.44
24.95	34.38	25.18	59.33	8.97
25.74	17.06	18.67	42.8	5.46
29.17	38.59	29.63	67.76	8.5
44.58	32.3	35.86	76.98	5.16
38.86	44.08	37.65	82.94	7.64
47.06	67.51	52.11	114.57	10.35

Average ratio of column 5 to column 4: .1358
Average correlation coefficient: .3746

Figure 2

Two other factors contribute to shortening the run time necessary to compare policies. First, each simulation is begun at the state where a substitution is made in one system and not in the other and continued only until they reach the same state. In fact the results obtained could have been achieved by simulating the entire process for the two policies. If we were to do this, there would be no difference in backorder days until the systems reach the state where the substitution is performed or not and the differences would cease when the systems reach the same state. The simulation, then, can be regarded as a series of disjoint cycles, each beginning at the state in question. It suffices to simulate only these cycles.

Second, we make only one substitution of the kind in question for each run, but the policy we are evaluating requires a substitution each time the new state is reached. We justify this by appealing to the linear programming formulation and solution of the problem. In practice the strategies for an optimal solution are pure, but for linear programming, the feasible solutions are mixed strategies on the states. In the algorithm an entering variable is selected by comparing the rate of change of the objective function with respect to a small change in the variable. A small change from 0 in a control variable corresponds in a probabilistic setting to making a substitution with a very small probability. To compare policies by simulation, if this substitution were made with the same small probability, most runs would consist of single substitutions at the state in question since, even if this state were reached before the systems came back

together, the probability of a second substitution would be very small. In the simulation, the effect of not making a second substitution tends to be to bring the systems together sooner and shorten the run length.

We considered but did not use variance reducing techniques such as antithetic variables and stratified sampling. Our intuitive feelings were that the large number of events in each run would make questionable their benefits and that the computational complexity would not be justified.

V. Summary and Comments Concerning Future Work

We have developed an efficient method for comparing policies which differ at just one state. This method appears to be computationally superior to other methods we have investigated in two respects: the amount of computer time does not grow as rapidly with increased system size and, perhaps even more relevant in terms of absolute limitations, the simulations do not use the large amounts of storage required to solve linear programming problems or systems of equations.

Continued investigation of the structure of optimal policies has reduced the number of states necessary to consider for inclusion in an optimal policy [2]. The development of a scaling technique allows large problems to be approximated by small problems that can be solved explicitly and the extrapolated solution used as a first approximation for a search with simulation.

We are currently investigating other models for the interchangeability/substitutability problem and are able to extend many of the techniques developed for this model to multi-echelon and multi-item problems [3].

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A PROCEDURE FOR FINDING A NEARLY OPTIMAL
POLICY FOR THE INVENTORY SYSTEM WITH
SEVERAL INTERCHANGEABLE RECOVERABLE ITEMS

by

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Table of Contents

1. The Multiple-Item Single-Location Inventory System	1
2. A Procedure for Obtaining an Approximation to the Optimal Substitution Policy	3
3. The Form of an Optimal Substitution Policy	8
4. An Example	11
5. Areas of Further Investigation	17

Abstract

In this paper we examine the substitutability problem for several recoverable items that fail at a single location. We assume the failure processes for each type of item are independent, stationary, Poisson processes. We also assume the repair times are independent and exponentially distributed. Furthermore, we assume the system is a closed system; that is, no items are added to or deleted from the system. We develop a procedure for obtaining an approximation to an optimal substitution policy for this system. The procedure uses a sequence of two-item inventory systems to model the multiple-item system. We then examine several properties that we expect an optimal substitution policy to possess, and show through an example how to use these properties to describe a comprehensive optimal substitution policy. Finally we identify some areas suggested by this problem which merit further investigation.

1. The Multiple-Item Single-Location Inventory System

In this section we present a single location inventory system with several types of items: type 1 through type n. The assemblies in which these items are installed are called units; each unit contains only one item. We assume that the units themselves are also classified as being of one of the types from type 1 to type n. When a unit of type k fails, it can be repaired with any serviceable item of type k or greater. Thus there is a one-directional substitutability among the items; an item of type k may be used whenever an item of a lesser type is used.

For example, there may be three units corresponding to successive generations of a computer. The three items might be the successive generations of a circuit board installed as original equipment on the three generations of the computer. Each version of the circuit board is designed to be used in the current generation of the computer and in all previous generations, but is incompatible with the newer generations of the computer. In this example, a second-generation circuit board may be used in both the first and second-generation computers, and a second-generation computer may be repaired with either a second or third-generation circuit board.

Let N_i be the number of units of type i , and let M_i be the number of spare items of type i . Then the total number of items of type i is $(N_i + M_i)$. Let n_{ij} be the number of type i items installed in type j units, and let m_i be the number of serviceable type i items in spare stock.

According to the substitution rules we have established, n_{ij} is zero whenever i is less than j . The number of items of type i in

repair is given by $(N_i + M_i - m_i - \sum_{j=1}^i n_{ij})$, and the number of back-orders associated with type j units is $(N_j - \sum_{i=j}^n n_{ij})$.

We suppose that the failure times of type i items installed in type j units are independent, exponentially distributed random variables with mean $1/\lambda_{ij}$. The repair times of type i items are independent, exponentially distributed random variables with mean $1/\mu_i$; we assume the repair time does not depend on the type of unit in which the item is installed when it fails.

We suppose that no cannabilization occurs in the inventory system, so that an item remains in the unit in which it is installed until it fails. We also suppose that no condemnations occur, so that every item is repaired and returned to service.

The measure of performance of the inventory system is the expected number of backorders. It seems clear that an optimal policy would never allow a backorder for a unit of type i when there is a spare item of type i in stock. But it is not at all obvious whether to substitute an item of a type greater than i or to allow a backorder when there are no spare items of type i in stock. This decision will depend on the substitutability rules we have described, on the failure rates and repair rates, and on the number of items already in use or in repair. For example, repairing a unit of type j with an item of type i may result in subsequent avoidable shortages if it happens that a type j item becomes available and a type i unit fails. It may be advantageous to reserve some of the more useful items so that they will be available if they are needed in a newer generation unit.

2. A Procedure for Obtaining an Approximation to the Optimal Substitution Policy

This section develops a procedure for obtaining an approximation to the optimal substitution policy for the inventory system just described. It begins by recalling the solution for the two-item system developed by Heath, Muckstadt, and Shilepsky, and shows why a new procedure is needed to keep the problem down to manageable proportions. The motivation for the new procedure is given next, followed by a stage-by-stage accounting of how the procedure would be implemented.

To obtain an approximation to the optimal substitution policy for the multiple-item inventory system, we will rely heavily upon the analysis of the similar two-item inventory system by Heath, Muckstadt, and Shilepsky. They formulated the two-item inventory system as a continuous-time Markovian decision problem and showed how to find the optimal stationary Markovian control policy by using linear programming. The optimal substitution policy consists of a rule dictating whether to repair a type 1 unit with a type 2 item when no type 1 item is available. This rule depends on three factors: the number of backorders for type 1 units, the number of type 2 items in spare stock, and the number of type 2 items currently installed in type 1 units.

Unfortunately, the size of the linear program and the number of factors needed by the optimal substitution rule grow rapidly as the number of types of items increases, and this method of solution becomes impractical. For a three-item system, the optimal decision rule will depend on the net inventory position of each of three items together with the number of type 2 items already installed in type 1 units, the number of type 3 items

already installed in type 1 units, and the number of type 3 items already installed in type 2 units. For more than three items, the factors used by the optimal decision rule are even more numerous. So we must develop a new method or solution of the multiple-item problem for two reasons: the linear program used to solve the two-item problem becomes too large to be solved quickly and economically, and the optimal decision rule becomes unwieldy because it depends on too many factors.

To motivate the approach we will take in finding an approximate solution to the multiple-item inventory system, let us consider the inventory system from the perspective of the person who manages the warehouse which holds spare items of type i . His job is to receive serviceable spare items as they are repaired and to make available a spare item whenever the repair policy dictates that a repair should be made with an item of type i . This manager will be called upon to make a decision when he holds some spare items and an item fails in a unit of type j , where j is less than or equal to i . His decision will be either to make an item available for use or to hold his spare items in reserve.

We have already mentioned that any reasonable substitution policy will always call for the repair of a type i unit if a type i item is available. So the only decision which merits consideration is whether to make available a type i item to a type j unit, where j is less than i . When an item is made available to a type j unit, there is one fewer item available to the type i units. So the inventory manager will need to be concerned with the duration of time for which the item is on loan to a type j unit and the likelihood that he would need the item for a type i unit during the period of the loan.

Here we make a simplifying assumption which allows us to reduce the number of factors needed by an optimal decision rule. We will assume that the failure rates λ_{ij} are identical for all items i in all units j , and that the repair rates μ_i are identical for all items i . Thus the manager of spare items of type i will not care what type of unit is in need of repair. Whether it be a type 1 unit or a type $(i-1)$ unit, one backorder will be filled and one type i item will be unavailable for a period encompassing a failure with rate λ and a repair with rate μ .

This simplifying assumption enables the inventory manager to regard the demand for spare items as arising from only two sources: from units of type i and from units of type less than i . This means that we have a two-item system: type I is an aggregate comprised of all types less than i and type II is identical to type i . We already know how to find an optimal decision rule for a two-item inventory system. We will be able to use a sequence of two-item systems to find an optimal decision rule for the multiple-item system if we can adequately model the behavior of the aggregated system comprised of all types less than i .

Let us focus our attention on the aggregated system. It consists of items and units of types 1 through $(i-1)$. Units fail and are repaired with items according to some decision rule. Sometimes the decision rule dictates that an item not be repaired; sometimes there are no suitable spare items available for the repair. In these instances, a backorder exists in the aggregated system, and a demand is placed for a spare item of type i . The interaction between the aggregated system of types 1 through $(i-1)$ and the system of type i is in these demands placed when a backorder exists in the aggregated system, so the essential feature

we must capture in a model of the aggregated system is the distribution of backorders in the system.

The steady-state distribution of backorders in a single-item system can be computed analytically. The steady-state distribution of backorders in a two-item system can be obtained from the two-item model of Heath, Muckstadt, and Shilepsky, where a linear program is used to compute the steady-state probabilities of the continuous-time Markov chain model of the system. We propose to replace a two-item system with the single-item system which most closely matches the backorder distribution of the two-item system. We will enumerate the single item systems for all reasonably small numbers of units N and spare items M , compute the mean and the variance of the number of backorders expected in each system, and choose as the best match the system which most closely approximates the mean and the variance of the number of backorders expected in the two-item system.

With this technique we can recursively model an inventory system with any number of types of items. We will begin by replacing types 1 and 2 with the single-item system which most closely matches the backorder distribution of the two-item system. We will then consider the resulting single-item system along with the type 3 system to be another two-item system which we will replace with yet another single-item system. We will continue in this manner until we have included all types of items in the inventory system.

A stage-by-stage description of this procedure follows.

Stage 1. Consider only types 1 and 2. Use the approach described by Heath, Muckstadt, and Shilepsky to find the optimal decision rule for using type 2 items in type 1 units.

Stage 2. Add type 3 to the inventory system.

- (a) Aggregate types 1 and 2. The inventory manager for type 3 will receive demands for spare parts from units of types 1 and 2. Use the results of Stage 1 to determine the distribution of total backorders in units of types 1 and 2 under the optimal decision rule found in Stage 1.
- (b) Find a single-item inventory system which incurs backorders according to the same distribution as that of the total backorders in units of types 1 and 2. Denote this single-item system as type (1,2).
- (c) Use the approach of Heath, Muckstadt, and Shilepsky to find the optimal decision rule for using type 3 items in type (1,2) units.

Stage 3. Add type 4 to the inventory system.

- (a) Use the results of Stage 2c to determine the distribution of total backorders in units of types (1,2) and 3 under the optimal decision rule found in Stage 2c.
- (b) Find a single-item inventory system, type (1,2,3), which incurs backorders with the same distribution as that of the total backorders in units of types (1,2) and 3.
- (c) Find the optimal decision rule for using type 4 items in type (1,2,3) units.

Stages 4 through (n-1). Continue in this manner until type n has been added to the inventory system.

The results of this sequence of decision rules for two-item systems can be translated into an explicit rule for whether to repair a type j unit with a type i item. But this matter is not entirely straightforward. A discussion of the considerations involved follows in the next section.

3. The Form of an Optimal Substitution Policy

This section discusses the form of an optimal substitution policy for the multiple-item inventory system. It begins by reviewing the structure present in the optimal decision rule for a two-item system and describes some additional properties which an optimal decision rule for a multiple-item system should possess. Then we show how to use these properties in conjunction with the results of the sequence of two-item inventory systems discussed in the previous section to state explicitly the decision rule to follow in the multiple-item system.

Heath, Muckstadt, and Shilepsky examined the form of an optimal policy for the two-item inventory system. They showed that the states of the system could be described by three parameters.

$$\text{State} = (s_1, s_2, n_{21})$$

where s_1 = net inventory of type 1 items $(-N_1 \leq s_1 \leq M_1)$
 s_2 = net inventory of type 2 items $(-N_2 \leq s_2 \leq M_2)$
 n_{21} = number of type 2 items installed in type 1 units
 $(0 \leq n_{21} \leq \min\{N_1, M_2\})$.

A decision rule specifies a subset of states from which a type 2 item should be substituted into a type 1 unit. For a decision rule to be feasible, these states must correspond to backorders among type 1 items and spares among type 2 items. That is, the subset S must satisfy

$$S \subseteq \{(s_1, s_2, n_{21}) | s_1 < 0 \text{ and } s_2 > 0\}.$$

Every such subset corresponds to a control policy. However, there are three monotonicity relationships that one might expect the optimal subset to have.

- (1) $(s_1, s_2, n_{21}) \in S_{\text{opt}}$ and $s < s_1$ implies $(s, s_2, n_{21}) \in S_{\text{opt}}$.

That is, as the number of type 1 backorders increases, there should be a greater willingness to substitute.

- (2) $(s_1, s_2, n_{21}) \in S_{\text{opt}}$ and $s > s_2$ implies $(s_1, s, n_{21}) \in S_{\text{opt}}$.

That is, as the number of type 2 spares increases, there should be a greater willingness to substitute.

- (3) $(s_1, s_2, n_{21}) \in S_{\text{opt}}$ and $n < n_{21}$ implies $(s_1, s_2, n) \in S_{\text{opt}}$.

That is, as the number of type 2 items installed in type 1 units decreases, there should be a greater willingness to substitute. The explanation for this is that a smaller value of n_{21} means that more of the type 2 items on loan are in repair, and so are closer to being available again as spare items.

In our procedure which models the multiple-item inventory system as a sequence of two-item systems, we get a set of rules specifying whether to perform a repair or to hold the spare items in reserve. However, it is not clear which repair is to be performed. As an

illustration of the ambiguity involved here, consider the stage in which we solve a two-item system where type I is an aggregate of types 1 and 2, and type II represents type 3. Suppose that the optimal policy calls for substitution when $(s_1 = -2, s_2 = 1, \text{ and } n_{21} = 1)$. It seems reasonable to interpret $(s_1 = -2)$ to mean that there are two backorders within the type 1 and type 2 systems, $(s_2 = 1)$ to mean that there is one spare type 3 item, and $(n_{21} = 1)$ to mean that there is one type 3 item installed in a unit of types 1 or 2. But given this, which of the two backorders are we supposed to fill with the one spare item? To help us decide, we conjecture that the choice of which repair to make will obey an additional set of monotonicity relations.

- (1) If a substitution is called for and there is a spare item of type i and there are backorders of types j_1 and j_2 , where $j_1 < j_2 \leq i$, do not fill the backorder of type j_1 . The explanation is that this backorder would be easier to fill later.
- (2) If a substitution is called for and there are spare items of types i_1 and i_2 and a backorder of type j , where $j \leq i_1 < i_2$, do not use the item of type i_2 . The explanation is that this item is potentially more useful for filling subsequent backorders.

The principle behind both of these properties is to fill any backorder as inexpensively as possible, saving as much flexibility as possible for the future.

A consequence of these monotonicity properties is that care must be taken in translating the optimal policies for the sequence of two-item systems into a comprehensive policy for the multiple-item system. To illustrate, let us suppose that the optimal policy from Stage 1

(where type I is type 1 and type II is type 2) calls for no substitution when $(s_1 = -1, s_2 = 1, \text{ and } n_{21} = 0)$. Suppose also that the optimal policy from Stage 2 (where type I is types 1 and 2 and type II is type 3) calls for substitution when $(s_1 = -1, s_2 = 1, \text{ and } n_{21} = 0)$. What should we do if there is a type 1 backorder, a type 2 spare, and a type 3 spare, and there are no items currently on loan? The Stage 1 policy says not to substitute, so there is a backorder in the aggregated system comprised of types 1 and 2. The Stage 2 policy says to substitute a type 3 item in this set of circumstances. But it would violate a monotonicity property if we were to fill a type 1 backorder with a type 3 item when a type 2 item is available.

The resolution of this problem lies in separating the question of whether to fill a backorder from the question of which backorder to fill. The optimal policies for the sequence of two-item models should determine only whether to fill some existing backorder. The monotonicity properties should be used to determine which backorder to fill.

4. An Example

This section presents an example of a three-item inventory system. The steps taken in solving this problem as a sequence of two-item systems are outlined. The results are then used together with the monotonicity properties in formulating a policy which dictates whether to repair a failed unit and specifies the type of item to be used in the repair.

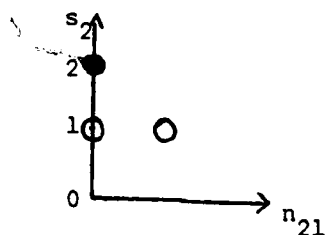
We begin with an inventory system with units and items of types 1, 2 and 3. Each unit requires the installation of one working item to be in working condition. Type 1 items may be used only in type 1 units; type 2 items may be used in units of types 1 and 2; type 3 items may be used in any of the units.

There are four units of each type and two spare items of each type; that is, there are four units and six items of each type. The times until failure of any item installed in a unit are assumed to be independent, identically-distributed exponential random variables having a mean of one time period. The repair times are also assumed to be independent, identically-distributed exponential random variables having a mean of one time period.

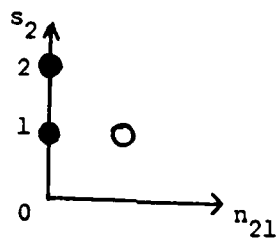
For the first stage, we consider only types 1 and 2. Following the procedure developed by Heath, Muckstadt, and Shilepsky, an optimal substitution strategy for this two-item problem was computed by the mathematical programming package MPSX. This strategy is presented below in the form of charts. Recall that we established that the state of the two-item inventory system could be characterized by three variables: the net inventory of type 1 items, the net inventory of type 2 items, and the number of type 2 items correctly installed in type 1 units. A substitution policy can then be characterized by a decision of whether to substitute a type 2 item into a type 1 unit for each state where the net inventory of type 1 items is negative and the net inventory of type 2 items is positive. The following charts depict those states where a substitution should be performed.

Optimal substitution policy for types 1 and 2

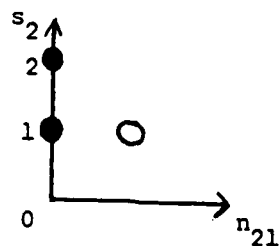
$s_1 = -1$



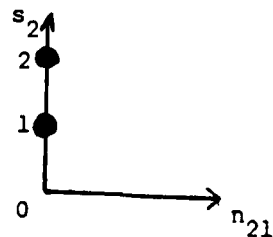
$s_1 = -2$



$s_1 = -3$



$s_1 = -4$

Key s_1 - net inventory of type 1 items s_2 - net inventory of type 2 items n_{21} - number of type 2 items installed in type 1 units

● - substitute a type 2 item into a type 1 unit

○ - make no repair

We can give a verbal characterization of this policy. "Substitute a spare type 2 item into a broken type 1 unit whenever there are two available spare items or whenever there are at least two broken type 1 units and no type 2 items are currently being used in type 1 units."

Stage 2 calls for the aggregation of types 1 and 2 as a single inventory system and the introduction of the type 3 system. We must first find a single-item system which behaves as the combined type 1 and 2 system. To this end, we use the steady-state probabilities computed by MPSX to determine the mean and the variance of the number of backorders using the optimal policy just described for the combined type 1 and 2 system. We compute

$$\text{Exp}[\text{backorders}] = 2.151$$

$$\text{Var}[\text{backorders}] = 2.206$$

We next search for a single-item inventory system which closely matches this mean and variance. Fortunately, the mean and variance of backorders in any single-item system can be readily calculated by computer. To keep the number of states from becoming too large for efficient computations, we restrict the number of units and the number of spare items to be no more than ten. The systems which best match the mean and variance of backorders are listed below.

Number of Units	Number of Spare Items	Mean of Backorders	Variance of Backorders
7	3	2.042	2.163
8	4	2.062	2.490
7	2	2.513	2.131
6	2	2.024	1.812
8	3	2.524	2.524

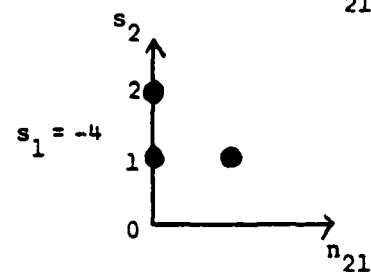
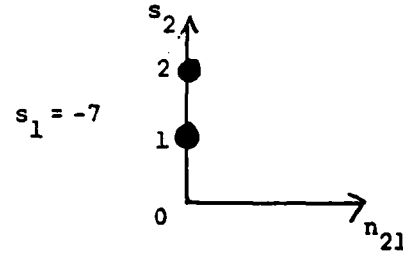
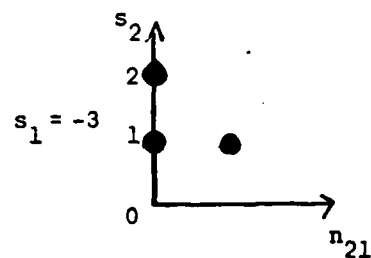
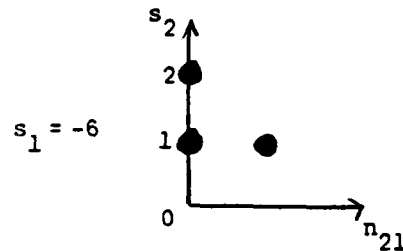
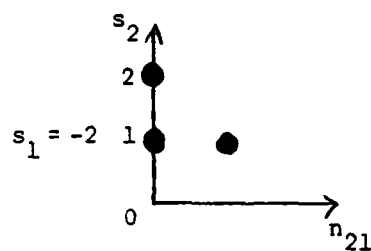
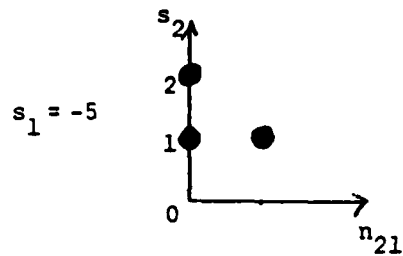
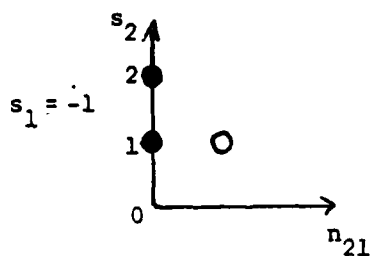
We chose to use the system with seven units and three spare items. We again turned to MPSX to find an optimal strategy for the two-item system, where type I (corresponding to types 1 and 2 combined) had seven units and three spare items and type II (corresponding to type 3) had four units and two spare items. The following charts depict the optimal substitution strategy.

A verbal characterization of this policy is to "substitute a spare type II item into a broken type I unit except when there is only one broken type I unit and another type I unit currently contains a type II item." Under this policy we expect the number of backorders to have a mean of 3.081 and a variance of 3.362, compared with a mean of 3.255 and a variance of 3.249 for the same inventory system using no substitution.

We have now solved the sequence of two-item systems and are ready to use the monotonicity properties to develop a comprehensive strategy for the three-item system. We first note that there are several sets of circumstances where the results from Stage I prescribe no substitution while the results from Stage II prescribe a substitution. One such situation is one type 1 backorder, one type 2 spare part, and one type 3 spare part, with no items on loan to a unit of a different type. In such situations, we should make the substitution but use the monotonicity properties and fill the backorder with the type 2 item instead of with the type 3 item.

There is a very appealing rationale for interpreting the results of the two stages in this manner. The decision not to substitute made in Stage I was made without any knowledge of the type 3 system.

Optimal substitution policy for types (1 and 2) and type 3.



Key

s_1 - net inventory of type I items

s_2 - net inventory of type II items

n_{21} - number of type II items
installed in type I units

● - substitute a type II item
into a type I unit

○ - make no repair

The decisions made at Stage I will necessarily be conservative because the policy is determined on the basis of no spare items being available from the type 3 system. When the potential availability of type 3 spare items is considered, we would expect to be willing to substitute items more freely. The type 3 inventory system can be said "to bankroll" the type 2 inventory system by providing available type 3 spare items to fill the extra backorders the type 2 system incurs as a result of its more liberal substitution policy.

So the optimal decision rule in this example will be to perform a substitution whenever it is dictated by either the Stage 1 or the Stage 2 decision rule. The decision about which backorder to fill should be made in accordance with the monotonicity properties, filling a backorder with the least versatile item able to fill that backorder.

5. Areas of Further Investigation

This section lists several areas of further investigation for matters that were either suggested or carefully avoided in the preceding sections.

(1) There is no analytical proof that the preceding procedure will produce an optimal substitution policy. A simulation study could be designed to evaluate the performance of the procedure we suggest here and compare it to the performance of other substitution strategies. The simulation study could also confirm that the monotonicity properties are indeed valid.

(2) There are variants of our dynamic programming approach which merit some consideration. Instead of building a sequence of two-item

systems from the bottom, we could start from the top and work down, matching the distribution of spare items in the aggregate system with a two-item system, instead of matching the distribution of backorders. Another alternative is to combine at each stage the pair of adjacent systems which would be expected to have the highest degree of substitution (resulting from system i incurring many backorders and system $(i+1)$ providing many spare items) on the grounds that a high degree of substitution causes a pair of systems to behave more as a single system. It would be interesting to see if these approaches produce similar optimal substitution policies.

(3) The criterion for choosing a single-item system to replace an aggregate system could be studied further. We might be able to produce a closer fit to the first two moments by varying the repair rate along with the number of units and the number of spare items in our search for an equivalent single-item system.

(4) The comprehensive substitution policy may be made more automated instead of using an ad hoc procedure based on the monotonicity properties. We illustrate this by continuing the example of the previous section. At Stage 2 an optimal policy was prescribed for substituting spare type 3 items into type 1 and 2 units. The type 3 system could have either zero, one, or two spare items. We could determine a comprehensive substitution policy by returning to Stage 1 to recalculate the optimal substitution policy as if the type 2 system had one and two additional spare items. In effect, we would be determining an optimal policy by explicitly incorporating the "bankroll" of spare type 3 items into the inventory of type 2 items.

(5) The monotonicity properties alone do not always enable us to choose which substitution to make. They do not, for example, tell us how to choose between repairing a type 1 unit with a type 2 item and repairing a type 3 unit with a type 4 item.